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APPLICATION OF SPLINES TO THE NUMERICAL SOLUTION OF TWO-POINT BOUNDARY-VALUE PROBLEMS

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20. ABSTRACT (Continued)

is described, and a source deck listing is included. Several sample problems solved by the program are presented.

PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC). The Air Force program manager was Elton R. Thompson. This work was done by ARO, Inc., AEDC Division (a Sverdrup Corporation Company), operating contractor for the AEDC, AFSC, Arnold Air Force Station, Tennessee, under ARO Project Number V32A-P1A. The manuscript was submitted for publication on August 23, 1978.

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1.0 INTRODUCTION

The availability of increased computer power within the past decade has spurred considerable effort toward development of computer programs for the numerical solution of the Navier-Stokes equations. The first and continuing effort involved numerical solution by finite-difference schemes. In recent years, however, other methods have come under investigation; two of these methods are finite elements and spline collocation. A preliminary test of spline collocation, namely application to numerically solve various two-point boundary-value problems, is the subject of the present report. Pertinent spline theory is developed from first principles. The spline collocation method, which is similar to that used by Rubin and Khosla (Ref. 1), is also fully developed. The problems considered are nonlinear, third- order, ordinary differential equations. A FORTRAN IV computer program to solve such problems is described, together with a source deck listing. Several example problems solved by the program are presented.

A spline interpolates between points by polynomials determined by various conditions of continuity. If a spline using polynomials of degree p is continuous along with its first q derivatives, $o \leqslant q \leqslant p$, then the spline is said to have a deficiency of $p - q$. For example, a cubic spline with continuous second derivatives has a deficiency of one. Rubin and Khosla (Ref. 1) have shown that classical three-point difference formulas are equivalent to a quadratic spline with a deficiency of zero. The method presented in this report uses a quintic spline of deficiency three.

Three-point, finite-difference methods, when applied to a linear equation, produce a tridiagonal algebraic system to be solved (Ref. 2), whereas spline collocation (for second-order equations) produces a tridiagonal system of 2 by 2 blocks. End conditions involving first and second derivatives are naturally imposed using spline collocation. For uniform grid spacing the three-point, finite-difference formula for the second derivative yields a second-order approximation, whereas the quintic spline gives third-order accuracy.

Splines were originally invented for interpolation, which means that interpolation of spline solutions is immediately available. This plus the suitability of spline collocation to handle nonuniform spacing makes it feasible to change grid spacing between iterations. This can be used effectively to improve accuracy of the resulting solution.

It is concluded that splines have many useful features and should have a role in future techniques for solving complicated problems.

2.0 SPLINE THEORY

Given points (x_i, y_i) with $i = 1, 2, ..., N$, a spline has the following properties:

1. It is a function defined on $[x_1, x_N]$.
2. It passes thru the points.
3. It is continuous. The splines considered in this report have continuous first and second derivatives.
4. It is defined by $(N - 1)$ polynomials, one for each interval $[x_i, x_{i+1}]$. The polynomials are determined by the conditions of continuity and the end conditions.

It is convenient for the succeeding analysis to make the following definitions.

$$h_i = x_{i+1} - x_i \tag{1}$$

$$\sigma_i = \frac{h_{i+1}}{h_i} \tag{2}$$

$$\eta_i = \frac{1}{h_i}(x - x_i) \tag{3}$$

$$\theta_i = 1 - \eta_i \tag{4}$$

Figure 1 shows the first two and last three points, (x_i, y_i), and their related h_i.

2.1 CUBIC SPLINES

A cubic spline is defined by

$$s(x) = p_i(x) \tag{5}$$

for $x_i \leqslant x \leqslant x_{i+1}$ with $i = 1, 2, ..., N - 1$ where the $p_i(x)$ are cubics. Since $s(x)$ passes thru the points,

$$s(x_i) = y_i \tag{6}$$

Since $s(x)$ is assumed to have continuous first and second derivatives, one can make the definitions

$$m_i = s'(x_i) \tag{7}$$

and

$$M_i = s''(x_i) \tag{8}$$

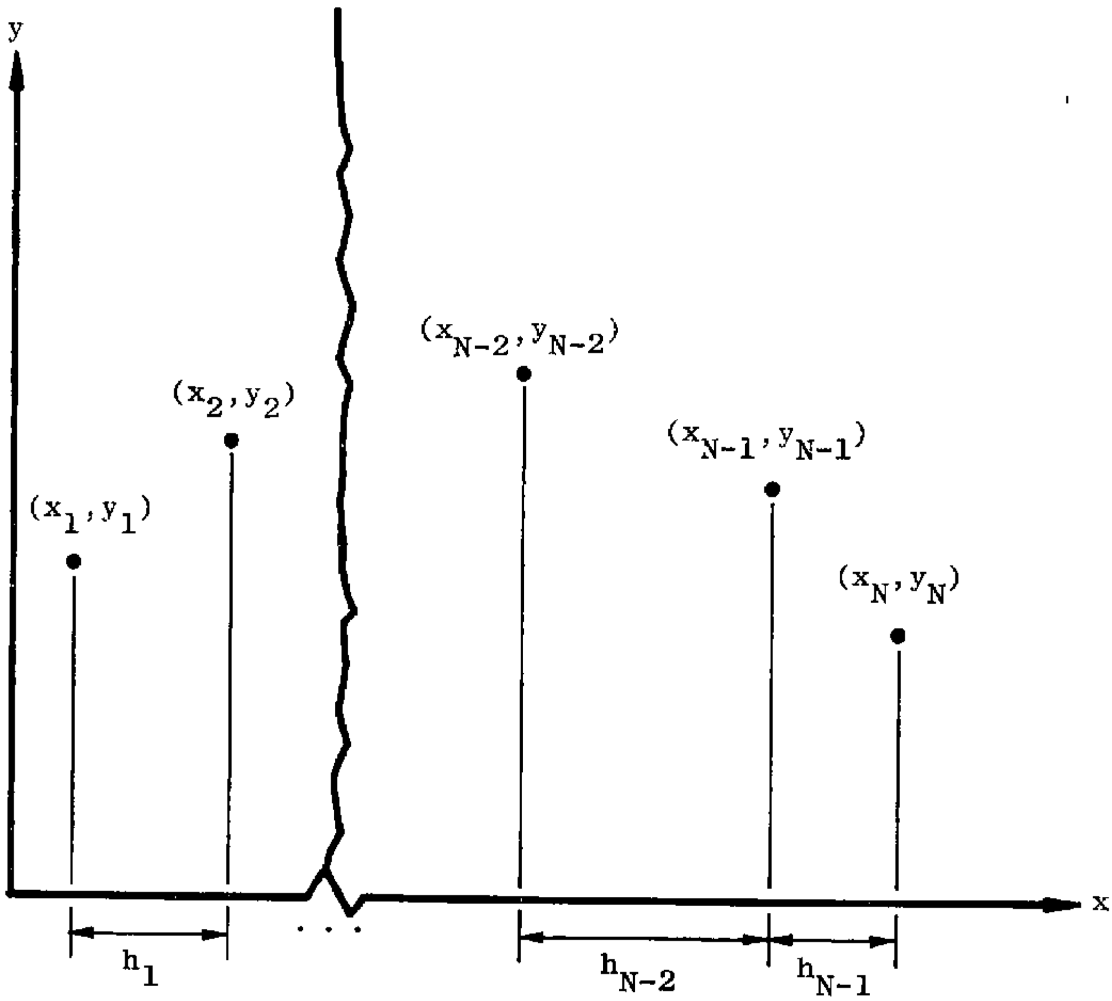

Figure 1. Clarification of the h_i.

The $p_i(x)$ can be written in terms of the m_i as follows. Define

$$y_{Ai} = y_i + \frac{1}{3} h_i m_i \tag{9}$$

and

$$y_{Bi} = y_{i+1} - \frac{1}{3} h_i m_{i+1} \tag{10}$$

then $p_i(x)$ can be written

$$p_i(x) = y_i \theta_i^3 + 3 y_{Ai} \eta_i \theta_i^2 + 3 y_{Bi} \eta_i^2 \theta_i + y_{i+1} \eta_i^3 \tag{11}$$

The first derivative is

$$p_i'(x) = \frac{3}{h_i} \Big[-y_i \theta_i^2 + y_{Ai} (\theta_i - 2\eta_i) \theta_i + y_{Bi} \eta_i (2\theta_i - \eta_i) + y_{i+1} \eta_i^2 \Big] \tag{12}$$

and the second derivative is

$$p_i''(x) = \frac{6}{h_i^2} \left[y_i \theta_i + y_{Ai} (\eta_i - 2\theta_i) + y_{Bi} (\theta_i - 2\eta_i) + y_{i+1} \eta_i \right] \tag{13}$$

The $p_i(x)$ are cubics in x, and it can be confirmed that, as defined,

$$p_i(x_i) = y_i \tag{14}$$

$$p_i(x_{i+1}) = y_{i+1} \tag{15}$$

$$p_i''(x_i) = m_i \tag{16}$$

and

$$p_i'(x_{i+1}) = m_{i+1} \tag{17}$$

The $p_i(x)$ have been written in terms of the m_i which, as yet, have not been determined. A relation among the m_i can be derived by the requirement that the second derivative be continuous; thus,

$$p_{i-1}''(x_i) = p_i''(x_i) = M_i \tag{18}$$

From Eq. (13),

$$p_{i-1}''(x_i) = \frac{6}{h_{i-1}^2} \left[y_{A,i-1} - 2 y_{B,i-1} + y_i \right] \tag{19}$$

Thus after substituting Eqs. (9) and (10) into (19) and simplifying,

$$M_i = \frac{2}{h_{i-1}} (2 m_i + m_{i-1}) - \frac{6}{h_{i-1}^2} (y_i - y_{i-1}) \tag{20}$$

Again, from Eq. (13),

$$p_i''(x_i) = \frac{6}{h_i^2}\left[y_i - 2y_{Ai} + y_{Bi}\right] \tag{21}$$

which, after substitution of Eqs. (9) and (10), yields

$$M_i = \frac{6}{h_i^2}(y_{i+1} - y_i) - \frac{2}{h_i}(2m_i + m_{i+1}) \tag{22}$$

Subtracting Eq. (22) from Eq. (20) and simplifying gives the desired relation among the m_i,

$$\frac{1}{h_{i-1}} m_{i-1} + 2\left(\frac{1}{h_{i-1}} - \frac{1}{h_i}\right) m_i + \frac{1}{h_i} m_{i+1}$$

$$= -\frac{3}{h_{i-1}^2} y_{i-1} + 3\left(\frac{1}{h_{i-1}^2} - \frac{1}{h_i^2}\right) y_i + \frac{3}{h_i^2} y_{i+1} \tag{23}$$

Assume end conditions

$$A_i y_i + B_i m_i + C_i M_i = D_i \tag{24}$$

for $i = 1$ and N where A_i, B_i, C_i, and D_i are given constants. In each case, $i = 1$ or $i = N$, at least one of the coefficients, B_i or C_i, must be nonzero. Since the y_i are given, the A_i are unnecessary and can be assumed zero. They were included for purposes of Section 3 on spline collocation where the y_i are unknown. With $i = 1$, substitution of Eq. (22) into (24) yields

$$\left(B_1 - \frac{4}{h_1} C_1\right) m_1 - \frac{2}{h_1} C_1 m_2$$

$$= D_1 - \left(A_1 - \frac{6}{h_1^2} C_1\right) y_1 - \frac{6}{h_1^2} C_1 y_2 \tag{25}$$

Similarly, with $i = N$, substitution of Eq. (20) into (24) yields

$$\left(B_N + \frac{4}{h_{N-1}} C_N\right) m_N + \frac{2}{h_{N-1}} C_N m_{N-1}$$

$$= D_N - \left(A_N - \frac{6}{h_{N-1}^2} C_N\right) y_N - \frac{6}{h_{N-1}^2} C_N y_{N-1} \tag{26}$$

Equations (25), (23) with $i = 2, 3, ..., N - 1$, and Eq. (26) form a tridiagonal system which can be solved for the m_i. When the solution for the m_i has been effected, the spline is known.

It is a straightforward matter to derive formulas for the integral of the spline. If one has a formula for

$$J_i(a, b) = \int_a^b p_i(x)\, dx \tag{27}$$

where $x_i \leqslant a < b \leqslant x_{i+1}$ then the integral over an arbitrary subinterval of $[x_1, x_N]$ can be obtained by summing the appropriate $J_i(a,b)$ with the proper arguments. When Eq. (11) is integrated,

$$J_i(a, b) = h_i \left[-\frac{1}{4} y_i \theta_i^4 - y_{Ai} \left(\eta_i + \frac{1}{4} \theta_i \right) \theta_i^3 \right.$$

$$\left. + y_{Bi} \eta_i^3 \left(\theta_i + \frac{1}{4} \eta_i \right) + \frac{1}{4} y_{i+1} \eta_i^4 \right]_a^b \tag{28}$$

Of particular interest is

$$J_i (x_i, x_{i+1}) = \frac{h_i}{4} \left[y_i + y_{Ai} + y_{Bi} + y_{i+1} \right] \tag{29}$$

When one has the given points, (x_i, y_i), and the calculated m_i, then the M_i can be computed from Eqs. (20) and (22). In analysis, in order not to give preferential treatment to one formula or the other at interior points, it is expedient to average the two equations to arrive at

$$M_i = \frac{1}{h_{i-1}} m_{i-1} + 2 \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) m_i - \frac{1}{h_i} m_{i+1}$$

$$+ \frac{3}{h_{i-1}^2} y_{i-1} - 3 \left(\frac{1}{h_{i-1}^2} + \frac{1}{h_i^2} \right) y_i + \frac{3}{h_i^2} y_{i+1} \tag{30}$$

The fundamental relation, Eq. (23), was found by subtracting Eq. (20) from (22). Equation (20) was obtained by adding Eqs. (20) and (22). So, in that sense, there is a symmetrical relation between Eqs. (23) and (30).

By eliminating the y_i from Eqs. (20) and (22), a relation between the m_i and M_i is obtained.

$$M_{i-1} + M_i = \frac{2}{h_i} (m_{i+1} - m_i) \tag{31}$$

The spline relations have been written in terms of unknown m_i. Analogous formulas in terms of unknown M_i can be derived. Replacing i with (i + 1) in Eq. (20) and eliminating m_{i+1} from Eqs. (20) and (22) yields

$$m_i = \frac{1}{h_i} (y_{i+1} - y_i) - \frac{h_i}{6} (2M_i + M_{i+1}) \tag{32}$$

Replacing i with (i - 1) in Eq. (22) and eliminating m_{i-1} from Eqs. (22) and (20) yields

$$m_i = \frac{1}{h_{i-1}} (y_i - y_{i-1}) + \frac{h_{i-1}}{6} (2M_i + M_{i-1}) \tag{33}$$

Averaging Eqs. (32) and (33) gives

$$m_i = -\frac{1}{2h_{i-1}} y_{i-1} + \frac{1}{2} \left(\frac{1}{h_{i-1}} - \frac{1}{h_i} \right) y_i + \frac{1}{2h_i} y_{i+1}$$

$$+ \frac{h_{i-1}}{12} M_{i-1} + \frac{1}{6} (h_{i-1} - h_i) M_i - \frac{h_i}{12} M_{i+1} \tag{34}$$

Subtracting gives

$$\frac{h_{i-1}}{6} M_{i-1} + \frac{1}{3}(h_{i-1} + h_i) M_i + \frac{h_i}{6} M_{i+1}$$

$$= \frac{-1}{h_{i-1}} y_{i-1} - \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) y_i + \frac{1}{h_i} y_{i+1} \tag{35}$$

The end conditions, Eq. (24), in terms of unknown M_i are found to be

$$\left(C_1 - \frac{h_1}{3} B_1\right) M_1 - \frac{h_1}{6} B_1 M_2$$

$$= D_1 - \left(A_1 - \frac{1}{h_1} B_1\right) y_1 - \frac{1}{h_1} B_1 y_2 \tag{36}$$

and

$$\left(C_N + \frac{h_{N-1}}{3} B_N\right) M_N + \frac{h_{N-1}}{6} B_N M_{N-1}$$

$$= D_N - \left(A_N + \frac{1}{h_{N-1}} B_N\right) y_N + \frac{1}{h_{N-1}} B_N y_{N-1} \tag{37}$$

As an alternative to solving for the m_i, Eqs. (36), (35) with i = 2, 3, ..., N - 1, and Eq. (37) form a tridiagonal system which can be solved for the M_i.

The preceding formulas on cubic splines are adequate for the present analysis. For a complete treatment and proofs, see Ref. 3.

2.2 TRUNCATION ERROR OF CUBIC SPLINE FORMULAS

Fyfe, Ref. 4, used operator methods to obtain a Taylor expansion of cubic spline relations with uniform spacing. His work can be generalized for unequal spacing as follows. Define the operator E_i by

$$E_i \, y_i = y_{i+1} \tag{38}$$

Let y(x) be a function such that $y(x_i) = y_i$. Indicate the jth derivative by $y^{(j)}(x)$ and define $y_i^{(j)} = y^{(j)}(x_i)$. For the purpose of this error analysis, the tenth derivative will be assumed continuous. Expanding the right-hand side of Eq. (38) in a Taylor series,

$$E_i \, y_i = y_i + h_i \, y_i' + \frac{1}{2} h_i^2 \, y_i'' + \ldots . \tag{39}$$

Using the operator D to denote differentiation, from Eq. (39),

$$E_i = 1 + h_i \, D + \frac{1}{2} h_i^2 \, D^2 - \ldots . \tag{40}$$

and it is interesting to observe that

$$E_i = \exp\,(h_i \, D) \tag{41}$$

where the exponential is defined by its expansion.

Since

$$E_{i-1}\, y_{i-1} = y_i \tag{42}$$

then (assuming existence)

$$y_{i-1} = E_{i-1}^{-1}\, y_i \tag{43}$$

and from Eq. (41),

$$E_{i-1}^{-1} = \exp\,(-h_{i-1}\, D) \tag{44}$$

Some consider the last step presumptuous; however, understanding that operator expressions are merely convenient representations of series, and assuming linearity, the distributive law, etc., rigorous justification is possible. This section will lean heavily on such algebraic analogies.

In operator notation, Eq. (23) becomes

$$\left[\frac{1}{h_{i-1}}\, E_{i-1}^{-1} + 2\left(\frac{1}{h_{i-1}} - \frac{1}{h_i}\right) + \frac{1}{h_i}\, E_i\right] m_i$$

$$= \left[-\frac{3}{h_{i-1}^2}\, E_{i-1}^{-1} - 3\left(\frac{1}{h_{i-1}^2} - \frac{1}{h_i^2}\right) + \frac{3}{h_i^2}\, E_i\right] y_i \tag{45}$$

and solving for m_i,

$$m_i = \frac{\left[-\frac{3}{h_{i-1}}\, E_{i-1}^{-1} + 3\left(\frac{1}{h_{i-1}^2} - \frac{1}{h_i^2}\right) + \frac{3}{h_i^2}\, E_i\right]}{\left[\frac{1}{h_{i-1}}\, E_{i-1}^{-1} + 2\left(\frac{1}{h_{i-1}} - \frac{1}{h_i}\right) + \frac{1}{h_i}\, E_i\right]}\; y_i \tag{46}$$

or, if one expands Eq. (46) into a series,

$$m_i = \Big[1 - \frac{1}{72}\,\sigma_{i-1}\,(\sigma_{i-1} - 1)\,(h_{i-1}\, D)^3$$

$$- \frac{\sigma_{i-1}}{180}\,\frac{\sigma_{i-1}^3 + 1}{\sigma_{i-1} + 1}\,(h_{i-1}\, D)^4$$

$$- \frac{1}{2{,}160}\,\sigma_{i-1}\,(\sigma_{i-1} - 1)\,(3\sigma_{i-1}^2 - 5\sigma_{i-1} + 3)\,(h_{i-1}\, D)^5 + \ldots\Big]\, y_i' \tag{47}$$

where σ_{i-1} is defined by Eq. (2). Thus it is found that m_i is a second-order-accurate approximation to y_i' (third-order error). For uniform spacing ($\sigma_{i-1} = 1$), it is third-order accurate.

In operator notation, Eq. (31) becomes

$$(E_i + 1)\ M_i = \frac{2}{h_i}\ (E_i - 1)\ m_i \tag{48}$$

so

$$M_i = \frac{2}{h_i}\ \tanh\left(\frac{1}{2}\ h_i\ D\right) mi \tag{49}$$

or

$$M_i = \frac{2}{h_i}\left[\left(\frac{1}{2}\ h_i\ D\right) - \frac{1}{3}\left(\frac{1}{2}\ h_i\ D\right)^3 + \frac{2}{15}\left(\frac{1}{2}\ h_i\ D\right)^5 - \frac{17}{315}\left(\frac{1}{2}\ h_i\ D\right)^7 - \ldots\right] m_i \tag{50}$$

Combining with Eq. (47) yields

$$\begin{aligned} M_i = \Big[1 &- \frac{1}{12}\ \sigma_{i-1}^2\ (h_{i-1}\ D)^2 - \frac{1}{72}\ \sigma_{i-1}\ (\sigma_{i-1} - 1)'\ (h_{i-1}\ D)^3 \\ &+ \frac{1}{360}\ \sigma_{i-1}\ (3\sigma_{i-1}^3 - 2\sigma_{i-1}^2 + 2\sigma_{i-1} - 2)\ (h_{i-1}\ D)^4 \\ &- \frac{1}{4{,}320}\ \sigma_{i-1}\ (\sigma_{i-1} - 1)\ (\sigma_{i-1}^2 - 10\sigma_{i-1} + 6)\ (h_{i-1}\ D)^5 + \ldots\Big]\ y_i'' \end{aligned} \tag{51}$$

Thus, M_i is a first-order-accurate approximation to y_i'' (second-order error). For uniform spacing ($\sigma_{i-1} = 1$) it is a second-order-accurate approximation.

Equations (47) and (51) do not agree exactly with Rubin and Khosla (Ref. 1); however, as Ref. 1 points out, the expression derived depends on which equation one begins with. This discrepancy is explained by the limited information conveyed by N points. The N points determine a polynomial of degree (N - 1), but there are infinitely many polynomials of higher degree which pass thru the points. Thus a Taylor expansion of the method is not unique. Considering the nonuniqueness, one may wonder about the value of the error analysis. The value comes from the conclusions drawn on the order of the method, which are valid and useful. From the error analysis, higher order approximations will be derived.

Substituting Eqs. (9) and (10) into (29) gives

$$J_i\ (x_i\ ,\ x_{i+1}) = \frac{1}{2}\ h_i\ (y_i + y_{i+1}) - \frac{1}{12}\ h_i^2\ (m_{i-1} - m_i) \tag{52}$$

This is seen to be the trapezoidal rule plus an extra term. In operator notation,

$$J_i\ (x_i\ ,\ x_{i+1}) = \frac{1}{2}\ h_i\ (E_i + 1)\ y_i - \frac{1}{12}\ h_i^2\ (E_i - 1)\ m_i \tag{53}$$

Substituting Eq. (47) into (53) and expanding gives

$$J_i (x_i , x_{i+1}) = \Bigg[1 + \frac{1}{2} h_i D + \frac{1}{6} (h_i D)^2 + \frac{1}{24} (h_i D)^3$$

$$+ \frac{1}{144} (h_i D)^4 + \frac{3\sigma_{i-1}^2 + 5\sigma_{i-1} - 5}{(12)(360)\sigma_{i-1}^2} (h_i D)^5$$

$$+ \frac{9\sigma_{i-1}^2 - 9\sigma_{i-1} + 4}{(12)(720)\sigma_{i-1}^3} (h_i D)^6 + \dots \Bigg] h_i y_i \tag{54}$$

Expanding y(x) in a Taylor series and integrating, one finds that

$$\int_{x_i}^{x_{i+1}} y(x)\, dx = \left[\sum_{j=0}^{\infty} \frac{(h_i D)^j}{(j+1)!} \right] h_i y_i \tag{55}$$

Comparing Eqs. (54) and (55),

$$J_i (x_i , x_{i+1}) = \int_{x_i}^{x_{i+1}} y(x)\, dx - \frac{1}{720} (h_i D)^5 \Bigg[1$$

$$+ \frac{3\sigma_{i-1}^2 - 5\sigma_{i-1} + 5}{6\sigma_{i-1}^2} h_i D$$

$$+ \frac{12\sigma_{i-1}^3 - 63\sigma_{i-1}^2 + 63\sigma_{i-1} - 28}{84\sigma_{i-1}^3} (h_i D)^2 + \dots \Bigg] D^{-1} y_i \tag{56}$$

Thus, the formula is found to be fourth-order accurate (fifth-order error).

2.3 HIGHER ORDER APPROXIMATIONS

From Eq. (51),

$$y_i'' = M_i + \frac{1}{12} \sigma_{i-1}^2 h_{i-1}^2 y_i^{iv} + \dots \tag{57}$$

If an approximation of zero-order accuracy (first-order error) were known for y^{iv}, then substitution into Eq. (57) would yield a second-order-accurate approximation for y''_i. Consider the conjecture

$$y_i^{iv} \approx \frac{2}{\sigma_{i-1}(\sigma_{i-1}+1)h_{i-1}^2}\left[\sigma_{i-1} M_{i-1} - (\sigma_{i-1}+1) M_i + M_{i+1}\right] \tag{58}$$

which is the three-point, finite-difference formula applied to the M_i. Substitution of Eq. (58) into Eq. (57) yields, as the approximation for y''_i,

$$\mathfrak{M}_i = M_i - \frac{\Delta_i}{6}\left[\sigma_{i-1} M_{i-1} - (\sigma_{i-1}+1) M_i + M_{i+1}\right] \tag{59}$$

where

$$\Delta_i = \frac{\sigma_{i-1}}{\sigma_{i-1}+1} \tag{60}$$

Note that if $\Delta_i = 0$ then $\mathfrak{M}_i = M_i$. In operator notation,

$$\mathfrak{M}_i = \left\{1 + \frac{\Delta_i}{6}\left[\sigma_{i-1} E_{i-1}^{-1} - (\sigma_{i-1}+1) + E_i\right]\right\} M_i \tag{61}$$

or, expanding the operator,

$$\mathfrak{M}_i = \left\{1 + \frac{\Delta_i}{6}\left[\frac{1}{2}\sigma_{i-1}(\sigma_{i-1}+1)(h_{i-1}D)^2 + \frac{1}{3!}\sigma_{i-1}(\sigma_{i-1}^2-1)(h_{i-1}D)^3 + \frac{1}{4!}\sigma_{i-1}(\sigma_{i-1}^3+1)(h_{i-1}D)^4 - \frac{1}{5!}\sigma_{i-1}(\sigma_{i-1}^4-1)(h_{i-1}D)^5 - \ldots\right]\right\} \mathring{M}_i \tag{62}$$

Substituting Eq. (51) into (62) gives

$$\begin{aligned}\mathfrak{M}_i = \Bigg\{1 &+ \frac{\sigma_{i-1}^2}{12}\left(\Delta_i \frac{\sigma_{i-1}+1}{\sigma_{i-1}} - 1\right)\left(h_{i-1}D\right)^2 \\ &+ \frac{\sigma_{i-1}(\sigma_{i-1}-1)}{72}\left[2\Delta_i(\sigma_{i-1}-1) - 1\right](h_{i-1}D)^3 \\ &+ \frac{\sigma_{i-1}}{720}\left[2(3\sigma_{i-1}^3 - 2\sigma_{i-1}^2 + 2\sigma_{i-1} - 2) - 5\Delta_i(\sigma_{i-1}^2-1)\right](h_{i-1}D)^4 \\ &- \frac{\sigma_{i-1}(\sigma_{i-1}-1)}{(30)(144)}\left[\Delta_i(\sigma_{i-1}+1)(4\sigma_{i-1}^2+5\sigma_{i-1}-6) - (\sigma_{i-1}^2 - 10\sigma_{i-1}+6)\right](h_{i-1}D)^5 \\ &+ \ldots\Bigg\} y''_i\end{aligned} \tag{63}$$

By design, when Δ_i is given by Eq. (60), $\mathcal{M}_i$ is a second-order-accurate approximation to y_i'' (third-order error) and for uniform spacing ($\sigma_{i-1} = 1$), it is third-order accurate. This is somewhat different from that of Rubin and Khosla, Ref. 1, who reached the same results but with

$$\Delta_i = \frac{\sigma_{i-1}^3 + 1}{\sigma_{i-1}(\sigma_{i-1} + 1)^2} \tag{64}$$

This discrepancy is not caused by error but is explained in the discussion following Eq. (51). Note for uniform spacing ($\sigma_{i-1} = 1$) that Eqs. (60) and (64) both give $\Delta_i = 1/2$. In the limit as σ_{i-1} approaches infinity, both equations give $\Delta_i = 1$; however, as σ_{i-1} approaches zero, Eq. (60) yields $\Delta_i = 0$, whereas Eq. (64) gives Δ_i = infinity.

In an analogous manner, a higher order approximation, $\mathcal{m}_i$ to y_i', can be obtained. Substituting Eq. (58) into Eq. (47), let

$$\mathcal{m}_i = m_i + \frac{\delta_i h_{i-1}}{36}\left[\sigma_{i-1} M_{i-1} - (\sigma_{i-1} - 1) M_i + M_{i+1}\right] \tag{65}$$

where

$$\delta_i = \frac{\sigma_{i-1} - 1}{\sigma_{i-1} + 1} \tag{66}$$

Note for uniform spacing ($\sigma_{i-1} = 1$) that $\delta_i = 0$ and $\mathcal{m}_i = m_i$. Therefore, $\mathcal{m}_i$ is an improvement over m_i only for nonuniform spacing. However, for uniform spacing, m_i was already third-order accurate. In the limit, when $\sigma_{i-1} = 0$, then $\delta_i = -1$, and when δ_{i-1} = infinity, then $\delta_i = 1$. In operator notation,

$$\mathcal{m}_i = m_i + \frac{\delta_i h_{i-1}}{36}\left[\sigma_{i-1} E_{i-1}^{-1} - (\sigma_{i-1} + 1) + E_i\right] M_i \tag{67}$$

Expanding the operator and substituting Eqs. (47) and (51) into Eq. (67) yields

$$\begin{aligned}\mathcal{m}_i = \Bigg\{ & 1 + \frac{\sigma_{i-1}}{72}\left[\delta_i(\sigma_{i-1} + 1) - (\sigma_{i-1} - 1)\right](h_{i-1} D)^3 \\ & + \frac{\sigma_{i-1}}{180}\left[5\delta_i(\sigma_{i-1}^2 - 1) - 6\,\frac{\sigma_{i-1}^3 + 1}{\sigma_{i-1} + 1}\right](h_{i-1} D)^4 \\ & - \frac{\sigma_{i-1}(\sigma_{i-1} - 1)}{4{,}320}\left[5\delta_i(\sigma_{i-1} + 1) + 2(\sigma_{i-1}^2 - 5\sigma_{i-1} + 3)\right](h_{i-1} D)^5 \\ & + \cdots\Bigg\} y_i'\end{aligned} \tag{68}$$

By design, when δ_i is given by Eq. (66), m_i is a third-order-accurate approximation to y_i' (fourth-order error).

The analysis on m_i and $\mathfrak{M}_i$ is valid only at interior points (i = 2, 3, ..., N - 1). At the end points (i = 1 and N) they will be defined as $m_i = m_i$ and $\mathfrak{M}_i = M_i$.

In the next section an approximation, K_i (x_i, x_{i+1}), to the integral of Eq. (55) will be derived as follows:

$$K_i(x_i, x_{i+1}) = \frac{1}{2} h_i (y_i + y_{i+1}) - \frac{1}{10} h_i^2 (m_{i+1} - m_i) + \frac{1}{120} h_i^3 (\mathfrak{M}_i + \mathfrak{M}_{i+1}) \tag{69}$$

It is convenient to include the error analysis of the formula at this time. In operator notation

$$K_i(x_i, x_{i+1}) = \frac{1}{2} h_i (E_i + 1) y_i - \frac{1}{10} h_i^2 (E_i - 1) m_i + \frac{1}{120} h_i^3 (E_i + 1) \mathfrak{M}_i \tag{70}$$

Expanding the operators and substituting Eqs. (68) and (63) into Eq. (70), one can arrive at

$$K_i(x_i, x_{i+1}) = \Big\{ 1 + \frac{1}{2} h_i D + \frac{1}{6} (h_i D)^2 + \frac{1}{24} (h_i D)^3 + \frac{1}{120} (h_i D)^4 + \left[\frac{1}{720} + \frac{(\sigma_{i-1} - 1)(2\sigma_{i-1} - 1)}{(36)(120)\sigma_{i-1}^2} \right] (h_i D)^5 + \left[\frac{1}{4{,}800} + \frac{16\sigma_{i-1}^3 + 5\sigma_{i-1}^2 + 95\sigma_{i-1} + 25}{(360)(120)\sigma_{i-1}^3} \right] (h_i D)^6 + \ldots \Big\} h_i y_i \tag{71}$$

Comparing Eqs. (71) and (55),

$$K_i(x_i, x_{i+1}) = \int_{x_i}^{x_{i+1}} y(x)\,dx \tag{72}$$

$$+ (h_i D)^6 \left\{ \frac{(\sigma_{i-1} - 1)(2\sigma_{i-1} - 1)}{(36)(120)\sigma_{i-1}^2} \right.$$

$$+ \left[\frac{16\sigma_{i-1}^3 + 5\sigma_{i-1}^2 + 95\sigma_{i-1} + 25}{(360)(120)\sigma_{i-1}^3} + \frac{1}{100{,}800} \right] (h_i D)$$

$$\left. + \ldots \right\} D^{-1} y_i$$

Thus Eq. (69) is found to be fifth-order accurate (sixth-order error) and for uniform spacing ($\sigma_{i-1} = 1$), it is sixth-order accurate. If m_i and $\mathfrak{M}_i$ were exact, then the formula would be sixth-order accurate.

2.4 QUINTIC SPLINES

Once the higher order approximations, m_i and $\mathfrak{M}_i$, have been found, a spline, z(x), is needed for which the first and second derivatives at x_i are m_i and $\mathfrak{M}_i$, respectively. That is,

$$z(x) = q_i(x) \tag{73}$$

for $x_i \leqslant x \leqslant x_{i+1}$, where the $q_i(x)$ are polynomials such that

$$z'(x_i) = m_i \tag{74}$$

and

$$z''(x_i) = \mathfrak{M}_i \tag{75}$$

for i = 1, 2, ..., N. For a particular $q_i(x)$, this requires the conditions

$$q_i(x_i) = y_i \tag{76}$$

$$q_i'(x_i) = m_i \tag{77}$$

$$q_i''(x_i) = \mathfrak{M}_i \tag{78}$$

$$q_i\,(x_{i+1}) = y_{i+1} \tag{79}$$

$$q_i'\,(x_{i+1}) = m_{i+1} \tag{80}$$

$$q_i''\,(x_{i+1}) = \mathfrak{M}_{i+1} \tag{81}$$

Since there are six conditions, $q_i(x)$ is quintic. Now one can define

$$y_{ai} = y_i + \frac{1}{5}\, h_i\, m_i \tag{82}$$

$$y_{bi} = y_i + \frac{2}{5}\, h_i\, m_i + \frac{1}{20}\, h_i^2\, \mathfrak{M}_i \tag{83}$$

$$y_{ci} = y_{i+1} - \frac{2}{5}\, h_i\, m_{i+1} + \frac{1}{20}\, h_i^2\, \mathfrak{M}_{i+1} \tag{84}$$

$$y_{di} = y_{i+1} - \frac{1}{5}\, h_i\, m_{i+1} \tag{85}$$

then

$$\begin{aligned} q_i\,(x) = {} & y_i\, \theta_i^5 + 5y_{ai}\, \eta_i\, \theta_i^4 + 10y_{bi}\, \eta_i^2\, \theta_i^3 \\ & + 10y_{ci}\, \eta_i^3\, \theta_i^2 + 5y_{di}\, \eta_i^4\, \theta_i + y_{i+1}\, \eta_i \end{aligned} \tag{86}$$

The first derivative is

$$\begin{aligned} q_i'\,(x) = {} & \frac{5}{h_i} \Big[-y_i\, \theta_i^4 + y_{ai}\, (\theta_i - 4\eta_i)\, \theta_i^3 + 2y_{bi}\, (2\theta_i - 3\eta_i)\, \eta_i\, \theta_i^2 \\ & + 2y_{ci}\, (3\theta_i - 2\eta_i)\, \eta_i^2\, \theta_i - y_{di}\, (4\theta_i - \eta_i)\, \eta_i^3 + y_{i+1}\, \eta_i^4 \Big] \end{aligned} \tag{87}$$

and the second derivative is

$$\begin{aligned} q_i''(x) = {} & \frac{20}{h_i^2} \Big[y_i\, \theta_i^3 + y_{ai}\, (3\eta_i - 2\theta_i)\, \theta_i^2 \\ & + y_{bi}\, (\theta_i^2 - 6\eta_i\, \theta_i + 3\eta_i^2)\, \theta i \\ & + y_{ci}\, (3\theta_i^2 - 6\eta_i\, \theta_i + \eta_i^2)\, \eta_i \\ & + y_{di}\, (3\theta_i - 2\eta_i)\, \eta_i^2 + y_{i+1}\, \eta_i^3 \Big] \end{aligned} \tag{88}$$

It can be verified that Eq. (86) does satisfy the six conditions, (76) thru (81).

To compute the integral of the quintic spline, if one has a formula for

$$K_i\,(a\,,\ b) = \int_a^b q_i\,(x)\; dx \tag{89}$$

where $x_i \leqslant a < b \leqslant x_{i+1}$, then one can obtain the integral over an arbitrary subinterval of $[x_1, x_N]$ by summing the appropriate $K_i(a,b)$ with the proper arguments. Integrating Eq. (86),

$$K_i(a, b) = h_i \Big[-\frac{1}{6} y_i \theta_i^6 - y_{ai} (\eta_i + \frac{1}{6} \theta_i) \theta_i^5$$

$$- y_{bi}\left(\frac{5}{2} \eta_i^2 + \eta_i \theta_i + \frac{1}{6} \theta_i^2\right) \theta_i^4$$

$$+ y_{ci}\left(\frac{5}{2} \theta_i^2 + \eta_i \theta_i + \frac{1}{6} \eta_i^2\right) \eta_i^4$$

$$+ y_{di} (\theta_i + \frac{1}{6} \eta_i) \eta_i^5 + \frac{1}{6} y_{i+1} \eta_i^6 \Big]_a^b \tag{90}$$

Of particular interest is

$$K_i(x_i, x_{i+1}) = \frac{1}{6} h_i (y_i + y_{ai} + y_{bi} + y_{ci} + y_{di} + y_{i+1}) \tag{91}$$

If one substitutes Eqs. (82) thru (85) into (91), then one obtains Eq. (69).

3.0 SPLINE COLLOCATION

The general solution to a second-order differential equation involves two arbitrary constants; thus two conditions must be made to determine a particular solution. If the two conditions are the specification of the values of the solution and its derivative at a particular x-value, then the problem is called an initial value problem, and one-step methods such as Runge-Kutta can be applied, with nonlinearity presenting no special problem. When the two conditions are given at two separate x-values, the problem is called a two-point boundary-value problem, and several techniques are applicable, all requiring iteration to handle nonlinearity. Some methods allow the problem to be left in its nonlinear form, for example shooting methods for which each iteration is an initial value problem. However, probably the best methods are those where the problem is linearized and each iteration solves a linear problem. That is the approach used in this report. Section 3.1 describes a spline collocation method to solve linear problems, and Section 3.2 shows how to linearize a problem and iterate to obtain a solution to the nonlinear problem.

A collocation method for solving a differential equation involves writing an approximate solution in terms of a number of parameters, then determining the parameters by requiring the approximate solution to satisfy the differential equation at certain discrete points. The approximate solution and the differential equation collocate

at those points. In the case of spline collocation, the approximate solution is the spline, and the parameters are the y_i and M_i. Since there are 2N unknowns, the Y_i and M_i, 2N equations are needed. Because the unknowns are spline parameters, N spline relations are in force. The other N equations come from the collocation requirements that the spline satisfy the differential equation at the x_i.

In the following analysis, quintic splines are assumed throughout; however, the equations for cubic splines are found by setting $\Delta_i = \delta_i = 0$. In Section 2, since the spline passed thru the points, z_i equaled y_i $[(z_i = z(x_i)]$. In this section, the requirement is for the spline to satisfy the differential equation at the x_i; therefore, z_i does not necessarily equal y_i. For this reason, when reference is made to equations in Section 2, it will be assumed that the y_i are replaced by z_i without further explanation.

3.1 THE LINEAR PROBLEM

Given the differential equation

$$y'' = \alpha(x) + \beta(x)\,y + \gamma(x)\,y' + \epsilon(x)\,I \tag{92}$$

where

$$I = I_1 + \int_{x_1}^{x} y(x')\,dx' \tag{93}$$

where I_1 is a given constant; and given end conditions, Eq. (24), it is desired to find a numerical solution for $x_1 \leqslant x \leqslant x_N$ by spline collocation.

Equation (92) should be classed as a third-order differential equation in I, since $y = I'$, $y' = I''$, and $y'' = I'''$. Orginally, the method to be presented was developed for second-order equations; later, it was extended to solve third-order equations by the above device. Spline collocation determines the spline, z(x), which satisfies the differential equation, Eq. (92), at the x_i. Therefore, for i = 1, 2, ..., N,

$$\mathfrak{M}_i = \alpha_i + \beta_i\, z_i + \gamma_i\, m_i + \epsilon_i\, g_i \tag{94}$$

where the subscript, i. on the functions α, β, γ, and ϵ indicates their evaluation at x_i. The g_i are defined by

$$g_1 = I_1 \tag{95}$$

and for i > 1,

$$g_i = g_{i-1} + K_{i-1}(x_{i-1}, x_i) \tag{96}$$

which by Eq. (69) becomes

$$g_i = g_{i-1} + \frac{1}{2} h_{i-1} (Z_i - Z_{i-1}) \tag{97}$$

$$- \frac{1}{10} h_{i-1}^2 (m_i - m_{i-1})$$

$$+ \frac{1}{120} h_{i-1}^3 (\mathfrak{M}_i + \mathfrak{M}_{i-1})$$

for $1 < i \leqslant N$. If one defines

$$Q_i = \sigma_{i-1} M_{i-1} - (\sigma_{i-1} + 1) M_i + M_{i+1} \tag{98}$$

then Eqs. (65) and (59) can be written

$$m_i = \begin{cases} m_i \,, & \text{for } i = 1 \text{ and } i = N \\ m_i + \dfrac{\delta_i h_{i-1}}{36} Q_i \,, & \text{for } 1 < i < N \end{cases} \tag{99}$$

$$\mathfrak{M}_i = \begin{cases} M_i \,, & \text{for } i = 1 \text{ and } i = N \\ M_i - \dfrac{\Delta_i}{6} Q_i \,, & \text{for } 1 < i < N \end{cases} \tag{100}$$

Ordinarily one would choose the z_i and m_i as the principal unknowns using Eqs. (23), (25), and (26). However, in light of Eq. (98) and its exclusive involvement of the M_i, the analysis can be simplified by choosing the z_i and M_i as the principal unknowns and using Eqs. (36), (35), and (37), which can be written

$$\left(A_1 - \frac{B_1}{h_1}\right) z_1 - \frac{B_1}{h_1} z_2 + (C_1$$

$$- \frac{1}{3} B_1 h_1) M_1 - \frac{1}{6} B_1 h_1 M_2 = D_1$$

$$\frac{1}{h_{i-1}} z_{i-1} - \left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) z_i + \frac{1}{h_i} z_{i-1}$$

$$- \frac{1}{6} h_{i-1} M_{i-1} - \frac{1}{3} (h_{i-1} + h_i) M_i - \frac{1}{6} h_i M_{i+1} = 0$$

$$\text{for } 1 < i < N$$

$$\left(A_N + \frac{B_N}{h_{N-1}}\right) z_N - \frac{B_N}{h_{N-1}} z_{N-1} + (C_N$$

$$+ \frac{1}{3} B_N h_{N-1}) M_N + \frac{1}{6} B_N h_{N-1} M_{N-1} = D_N \tag{101}$$

The m_i values come from Eqs. (32), (34), and (33), which can be written

$$m_i = \begin{cases} \dfrac{1}{h_1}(z_2 - z_1) - \dfrac{h_1}{6}(2M_1 + M_2) & \text{for } i = 1 \\[2ex] -\dfrac{1}{2h_{i-1}} z_{i-1} + \dfrac{1}{2}\left(\dfrac{1}{h_{i-1}} - \dfrac{1}{h_i}\right) z_i + \dfrac{1}{2h_i} z_{i+1} & \\[2ex] + \dfrac{h_{i-1}}{12} M_{i-1} + \dfrac{1}{6}(h_{i-1} - h_i) M_i - \dfrac{h_i}{12} M_{i+1} & \\ & \text{for } 1 < i < N \\[2ex] \dfrac{1}{h_{N-1}}(z_N - z_{N-1}) + \dfrac{h_{N-1}}{6}(2M_N + M_{N-1}) & \text{for } i = N \end{cases} \tag{102}$$

From Eqs. (94) thru (102) one could form a tridiagonal system with 7 by 7 blocks with unknowns z_i, M_i, m_i, Q_i, $\mathfrak{m}_i$, $\mathfrak{M}_i$, and g_i. However, for the sake of numerical efficiency, it is best to eliminate some of the unknowns before defining the system. The $\mathfrak{m}_i$ and $\mathfrak{M}_i$ can be eliminated by substitution of Eqs. (99) and (100) into Eqs. (94) and (97). Equation (94) becomes

$$\beta_i z_i - M_i - \gamma_i m_i - \epsilon_i g_i = -\alpha_i$$

$$\text{for } i = 1 \text{ and } i = N$$

$$\beta_i z_i - M_i + \gamma_i m_i + \epsilon_i g_i + \frac{1}{36}(\gamma_i \delta_i h_{i-1} - 6\Delta_i) Q_i = -\alpha_i \tag{103}$$

$$\text{for } 1 < i < N$$

If one defines

$$a_{1i} = \frac{h_{i-1}^3}{720}(\Delta_i - 2\delta_i) \tag{104}$$

and

$$a_{2i} = \frac{h_{i-1}^3}{720}\left(\Delta_i + \frac{2\delta_{i-1}}{\sigma_{i-2}}\right) \tag{105}$$

then, after m_i and $\mathfrak{M}_i$ are eliminated, the g_i can be written

$$g_i = \begin{cases} I_1 \text{ for } i = 1 \\ g_1 + \frac{1}{2} h_1 (z_2 + z_1) - \frac{1}{10} h_1^2 (m_2 - m_1) \\ + \frac{1}{120} h_1^3 (M_2 + M_1) + a_{12} Q_2 \text{ for } i = 2 \\ g_{i-1} + \frac{1}{2} h_{i-1} (z_i + z_{i-1}) - \frac{1}{10} h_{i-1}^2 (m_i - m_{i-1}) \\ + \frac{1}{120} h_{i-1}^3 (M_i + M_{i-1}) + a_{1i} Q_i + a_{2i} Q_{i-1} \text{ for } 2 < i < N \\ g_{N-1} - \frac{1}{2} h_{N-1} (z_N + z_{N-1}) - \frac{1}{10} h_{N-1}^2 (m_N - m_{N-1}) \\ + \frac{1}{120} h_{N-1}^3 (M_N + M_{N-1}) + a_{2N} Q_{N-1} \text{ for } i = N \end{cases} \tag{106}$$

The m_i and Q_i can be eliminated from the system by substitution of Eqs. (102) and (98) into Eqs. (103) and (106). The results, after substitution, can be called Eqs. (103)′ and (106)′, which will not be written out explicitly. The system then consists of Eqs. (106)′, (103)′, and (101), which can be written in the form

$$\begin{aligned} &D_{01} V_1 + D_{11} V_2 = R_1 \\ &D_{-12} V_1 - D_{02} V_2 + D_{12} V_3 = R_2 \\ &D_{-2i} V_{i-2} + D_{-1i} V_{i-1} + D_{0i} V_i + D_{1i} V_{i+1} = R_i \text{ for } 2 < i < N \\ &D_{-2N} V_{N-2} + D_{-1N} V_{N-1} + D_{0N} V_N = R_N \end{aligned} \tag{107}$$

where

$$V_i = \begin{bmatrix} g_i \\ M_i \\ z_i \end{bmatrix} \tag{108}$$

$$R_i = \begin{bmatrix} -I_1 \\ -\alpha_1 \\ D_1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -\alpha_i \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -\alpha_N \\ D_N \end{bmatrix}$$

$$\text{for } i=1 \qquad \text{for } 1 < i < N \qquad \text{for } i=N \tag{109}$$

and

$$D_{mi} = (d_{mijk}) \tag{110}$$

where all the d_{mijk} are zero, except for ones which will be listed below. A brief clarification of the subscripts may be useful. The subscript m takes values -2, -1, 0, and 1. The value 0 refers to the principal diagonal; -1 to the first lower diagonal; -2 to the second lower; and 1 to the upper diagonal. The subscript i indicates the ith point with $1 \leqslant i \leqslant N$. The subscripts j and k take values 1, 2, and 3. The j subscript refers to Eqs. (106)′, (103)′, and (101), respectively. The k subscript refers to g_i, M_i, and Z_i, respectively.

The nonzero d_{mijk} are as follows:

$$d_{-2i12} = \frac{h_{i-2}\,h_{i-1}^2}{120} + a_{2i}\,\sigma_{i-2} \quad \text{for } 2 < i \leq N \tag{111}$$

$$d_{-2i13} = -\frac{\sigma_{i-2}\,h_{i-1}}{20} \quad \text{for } 2 < i \leq N \tag{112}$$

$$d_{-1i11} = 1 \quad \text{for } 1 < i \leq N \tag{113}$$

$$d_{-1i12} = \begin{cases} -\dfrac{h_1^3}{30} + a_{12}\,\sigma_1 & \text{for } i = 2 \\[2ex] \dfrac{h_{i-1}^3}{60}\left(\dfrac{1}{\sigma_{i-2}} - 1\right) + a_{1i}\,\sigma_{i-1} - a_{2i}\,(\sigma_{i-2} + 1) & \text{for } 2 < i < N \\[2ex] \dfrac{h_{N-1}^3}{120}\left(\dfrac{2}{\sigma_{N-2}} - 3\right) - a_{2N}\,(\sigma_{N-2} + 1) & \text{for } i = N \end{cases} \tag{114}$$

$$d_{-1i13} = \begin{cases} \dfrac{9h_1}{20} & \text{for } i = 2 \\[2ex] \dfrac{h_{i-1}}{20}\,(\sigma_{i-2} + 10) & \text{for } 2 < i < N \\[2ex] \dfrac{h_{N-1}}{20}\,(\sigma_{N-2} + 11) & \text{for } i = N \end{cases} \tag{115}$$

$$d_{0i11} = -1 \quad \text{for } 1 \leq i \leq N \tag{116}$$

$$d_{0i\,12} = \begin{cases} \dfrac{h_1^3}{120}\,(2\sigma_1 - 3) - a_{12}\,(\sigma_1 + 1) & \text{for } i = 2 \\[2ex] \dfrac{h_{i-1}^3}{60}\,(\sigma_{i-1} + 1) - a_{1i}\,(\sigma_{i-1} + 1) + a_{2i} & \text{for } 2 < i < N \\[2ex] -\dfrac{h_{N-1}^3}{30} + a_{2N} & \text{for } i = N \end{cases} \tag{117}$$

$$d_{0i\,13} = \begin{cases} \dfrac{h_1}{20}\left(\dfrac{1}{\sigma_1} + 11\right) & \text{for } i = 2 \\[2ex] \dfrac{h_{i-1}}{20}\left(\dfrac{1}{\sigma_{i-1}} + 10\right) & \text{for } 2 < i < N \\[2ex] \dfrac{9h_{N-1}}{20} & \text{for } i = N \end{cases} \tag{118}$$

$$d_{1i\,12} = \frac{h_{i-1}^2\, h_i}{120} + a_{1i} \qquad \text{for } 1 < i < N \tag{119}$$

$$d_{1i\,13} = -\frac{h_{i-1}}{20\sigma_{i-1}} \qquad \text{for } 1 < i < N \tag{120}$$

$$d_{-1i\,22} = \begin{cases} \dfrac{\gamma_i\, h_{i-1}}{12} + \dfrac{\sigma_{i-1}}{36}\,(\gamma_i\,\delta_i\, h_{i-1} - 6\Delta_i) & \text{for } 1 < i < N \\[2ex] \dfrac{\gamma_N\, h_{N-1}}{6} & \text{for } i = N \end{cases} \tag{121}$$

$$d_{-1i\,23} = \begin{cases} -\dfrac{\gamma_i}{2h_{i-1}} & \text{for } 1 < i < N \\[2ex] -\dfrac{\gamma_N}{h_{N-1}} & \text{for } i = N \end{cases} \tag{122}$$

$$d_{0i\,21} = \epsilon_i \qquad \text{for } 1 \le i \le N \tag{123}$$

$$d_{0i22} = \begin{cases} -1 - \dfrac{\gamma_1 h_1}{3} & \text{for } i = 1 \\ -1 + \dfrac{\gamma_i}{6}\left(h_{i-1} - h_i\right) - \dfrac{1}{36}\left(\gamma_i \delta_i h_{i-1} - 6\Delta_i\right)\left(\sigma_{i-1} + 1\right) & \text{for } 1 < i < N \\ -1 + \dfrac{\gamma_N h_{N-1}}{3} & \text{for } i = N \end{cases} \tag{124}$$

$$d_{0i23} = \begin{cases} \beta_1 - \dfrac{\gamma_1}{h_1} & \text{for } i = 1 \\ \beta_i + \dfrac{\gamma_i}{2}\left(\dfrac{1}{h_{i-1}} - \dfrac{1}{h_i}\right) & \text{for } 1 < i < N \\ \beta_N + \dfrac{\gamma_N}{h_{N-1}} & \text{for } i = N \end{cases} \tag{125}$$

$$d_{1i22} = \begin{cases} -\dfrac{\gamma_1 h_1}{6} & \text{for } i = 1 \\ -\dfrac{\gamma_i h_i}{12} + \dfrac{1}{36}\left(\gamma_i \delta_i h_{i-1} - 6\Delta_i\right) & \text{for } 1 < i < N \end{cases} \tag{126}$$

$$d_{1i23} = \begin{cases} \dfrac{\gamma_1}{h_1} & \text{for } i = 1 \\ \dfrac{\gamma_i}{2h_i} & \text{for } 1 < i < N \end{cases} \tag{127}$$

$$d_{-1i32} = \begin{cases} -\dfrac{h_{i-1}}{6} & \text{for } 1 < i < N \\ \dfrac{B_N h_{N-1}}{6} & \text{for } i = N \end{cases} \tag{128}$$

$$d_{-1i33} = \begin{cases} \dfrac{1}{h_{i-1}} & \text{for } 1 < i < N \\ -\dfrac{B_N}{h_{N-1}} & \text{for } i = N \end{cases} \tag{129}$$

$$d_{0i32} = \begin{cases} C_1 - \dfrac{B_1 h_1}{3} & \text{for } i = 1 \\ -\dfrac{1}{3}(h_{i-1} + h_i) & \text{for } 1 < i < N \\ C_N + \dfrac{B_N h_{N-1}}{3} & \text{for } i = N \end{cases} \tag{130}$$

$$d_{0i33} = \begin{cases} A_1 - \dfrac{B_1}{h_1} & \text{for } i = 1 \\ -\left(\dfrac{1}{h_{i-1}} + \dfrac{1}{h_i}\right) & \text{for } 1 < i < N \\ A_N - \dfrac{B_N}{h_{N-1}} & \text{for } i = N \end{cases} \tag{131}$$

$$d_{1i32} = \begin{cases} -\dfrac{B_1 h_1}{6} & \text{for } i = 1 \\ -\dfrac{h_i}{6} & \text{for } 1 < i < N \end{cases} \tag{132}$$

$$d_{1i33} = \begin{cases} \dfrac{B_1}{h_1} & \text{for } i = 1 \\ \dfrac{1}{h_i} & \text{for } 1 < i < N \end{cases} \tag{133}$$

Figure 2 shows the coefficient matrix for N = 7. Each small square corresponds to a $d_{m\,ijk}$, and each large square corresponds to a $D_{m\,i}$. The X's mark the nonzero elements. The most straightforward way to solve the system is to use a banded system solver. The matrix is seen to have five diagonals above and below the principal diagonal for a bandwidth of 11.

The matrix can be treated as four diagonals of 3 by 3 blocks. The upper diagonal can be eliminated and the solution obtained by forward substitution. Some advantage can be taken of the sparseness of D_{-2i} (two nonzero elements). One must be careful in programming such a procedure if the procedure is intended to improve on the banded solver.

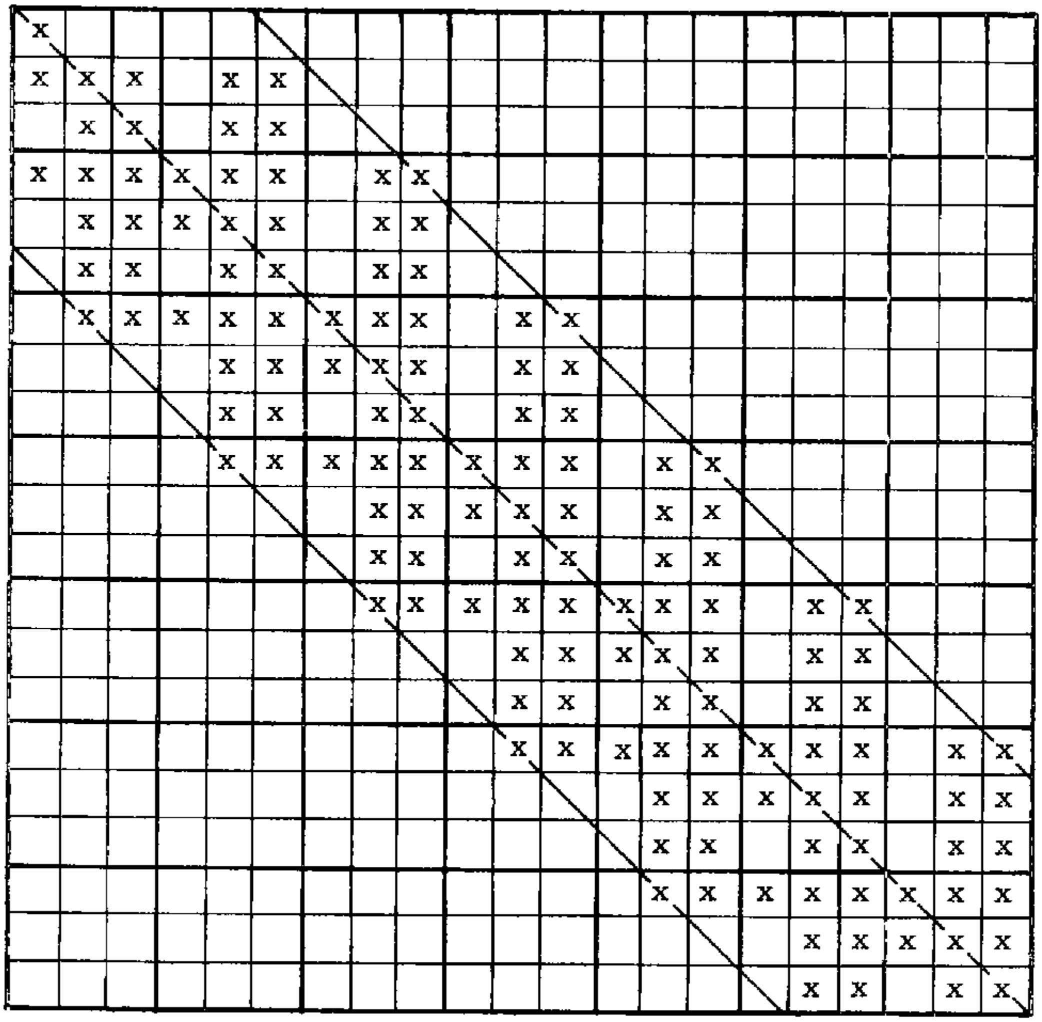

Figure 2. Coefficient matrix for N = 7.

When $\epsilon(x) \equiv 0$, then Eqs. (103) and (101) become uncoupled from Eq. (106) and a tridiagonal of 2 by 2 blocks can be solved for the M_i and z_i. The integral can then be computed from Eq. (69) if it is needed.

3.2 THE NONLINEAR PROBLEM

Most problems are nonlinear. Instead of Eq. (92) one may have

$$y'' = f(x\,,\ y\,,\ y',\ I) \tag{134}$$

Methods of solution of such problems often involve a process called quasi-linearization. The process requires an "in-hand" solution, which to begin with may be just a guess. Indicating the in-hand solution by a bar,

$$\begin{aligned} y'' \approx\ & f(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I}) + f_2(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I})\,(y - \bar{y}) \\ & + f_3(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I})\,(y' - \bar{y}') - f_4(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I})\,(I - \bar{I}) \end{aligned} \tag{135}$$

where f_j indicates the partial derivative of f with respect to the jth argument. Equation (135) can be put in the form of Eq. (92) where

$$\begin{aligned} \alpha(x) =\ & f(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I}) - f_2(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I})\,\bar{y} \\ & - f_3\,(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I})\,\bar{y}' - f_3(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I})\,\bar{I} \end{aligned} \tag{136}$$

$$\beta(x) = f_2\,(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I}) \tag{137}$$

$$\gamma(x) = f_3(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I}) \tag{138}$$

and

$$\epsilon(x) = f_4(x\,,\ \bar{y}\,,\ \bar{y}',\ \bar{I}) \tag{139}$$

With α, β, γ, and ϵ defined by Eqs. (136) thru (139), a solution can be obtained by the method of Section 3.1. This solution becomes the new in-hand solution. Iteration continues until a converged solution is obtained or until it becomes apparent that the method (for a particular case) fails to converge.

4.0 THE COMPUTER PROGRAM

The computer program was written in FORTRAN for the IBM 370/165 computer. To apply the program to a particular problem, the user must provide the functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$, and $\epsilon(x)$ of Eq. (92). This is done via a subroutine, QLDE. Only the text of the subroutine need be changed, the interface having already been programmed. The user must also provide the end condition constants of Eq. (24); the constant, I_1, of Eq. (93); a starting solution; and various logic control variables. These inputs are made in subroutine DATA. The different parameters are described by comments, and the user sets their values as desired. One other routine, CHEKBM, is problem dependent; however, it is not essential to the solution. CHEKBM is evoked after a solution has been found and the user can check or process the solution as desired. The use procedure of the computer program is demonstrated in Section 5 by example problems.

A listing of the program is given in Appendix A. A conscientious effort was made to make the program clean and modular, and to use comments liberally. Below are notes on some of the subroutines. If a subroutine is not included below, then it is adequately documented by comments. Line numbers refer to sequence numbers in columns 73 thru 80.

PROCED

The general logic of the program is contained in this routine. The DO loop of Line 340 provides the option of solving multiple problems. Exit from this loop is effected by a RETURN 2 from DATA (Line 350). Iteration is performed by the DO loop of Line 380 A converged solution (and branch-out of the loop) is indicated by a RETURN 1 from UPDATE (Line 430). The user is given the opportunity to check or process the solution as desired in CHEKBM, Line 530. The solution will be written on tape or disk, Line 540, or printed, Line 550, if the proper indicators (JUNIT and LPRNT) are set in DATA.

CHEKBM

The user can check or process the solution as desired in this routine. If no such need exists, then Lines 1770 thru 1960 can be deleted or, alternately, Line 530 of PROCED can be deleted.

CHEKDE

If the solution at the collocation points is the input arguments to this routine, then in effect the routine merely checks the validity of the solution from BANDED. This

routine can be used to see how well the solution satisfies the differential equation at points other than the collocation points. as follows:

1. Call QUINTS to determine the solution at the points of interest.

2. Call NTEGRL to compute the integral at the points.

3. Call QLDE to compute the α_i, β_i, γ_i, and ϵ_i.

4. Call CHEKDE and it checks the solution with the differential equation.

DATA

The logic control variables, end condition constants, and starting solution are input via this routine. Note the availability to UNIFRM to compute uniform spacing and GUESS to compute a quintic solution which satisfies the end conditions.

DELTAS

This routine evaluates Eqs. (60) and (66) for Δ_i and δ_i and has been made a separate routine with the idea that it might be useful to experiment with different values. It was pointed out that the expressions for Δ_i and δ_i are not unique. thus it would be interesting to determine their influence. Also, setting $\Delta_i = \delta_i = 0$ reduces the higher order quintic spline method to a cubic, so that the refinement of the method can be evaluated.

DIFRNC AND ERROR

The user should be aware of the availability of these routines which have been found useful in the various checking routines.

GUESS

GUESS is available to the user to compute a starting solution. It determines a quintic solution which is consistent with the end conditions. The method used by GUESS is presented in Appendix B.

NORM

NORM is used to check for convergence. The user has a choice of two tests indicated by the variable LNORM set in DATA. The first test finds the greatest difference between the two solutions relative to the range of the first solution. The second test finds the greatest difference between the two solutions relative to the solutions' local values. The second test is more stringent, especially for solutions which approach an asymptote.

RESPAC

RESPAC computes a new spacing with smaller steps where the solution changes more rapidly. The method used is presented in Appendix C. If no respacing is desired, then the parameter RSC can be set to a large value in DATA.

RESULT

The solutions returned by BANDED are the g_i, M_i, and z_i [See Eq. (108)]. RESULT computes $\mathfrak{m}_i$ (in the temporary variable, T1) from Eq. (102) and then m_i and $\mathfrak{M}_i$ from Eqs. (99) and (100).

SYSTEM

SYSTEM computes the d_{mijk} from Eqs. (111) thru (133).

PAGER

The user should replace Line 7560 with his own identification.

TIMCHK AND GETNOW

These routines are cosmetic rather than essential. Dummy routines are supplied in order to present a complete working program.

The user can replace Line 7800 with his own job identification or ideally would automate the routine to supply the job number, time, and date, as designed, via the mechanisms of his computer configuration.

5.0 EXAMPLE PROBLEMS

The computer program as listed in Appendix A solves the first example problem (Section 5.1). The program as listed will be called the reference program. Since most of the input must be programmed, for example the functions, $\alpha(x)$, $\beta(x)$, $\gamma(x)$, and $\epsilon(x)$, different problems are run by modification of the reference program.

The modifications to run the example problems are listed in Appendix D. They are presented in the format of an in-house update program which has three command statements:

```
-DEL   L1   [ , L2]
-INS   L1
-REP   L1   [ , L2]
```

The L1 and L2 are sequence numbers (i.e., the numbers in columns 73 thru 80 of the reference program). The brackets indicate that the inclusion of L2 is optional. The command -DEL deletes lines L1 thru L2. If L2 is omitted, then only line L1 is deleted. The command -INS inserts lines after line L1. The insertion lines follow the command statement and continue to the next command or else to the end of the data. The command -REP replaces lines L1 thru L2. If L2 is omitted, then only line L1 is replaced. The replacement lines follow the command statement and continue to the next command or to the end of the data.

At AEDC the reference program is kept on a permanent file. The operating procedure to run a particular problem is to create the modified program on a temporary file via the update program, then to compile and execute the modified program.

The printed output of the example problems is given in Appendix E.

5.1 EXAMPLE 1

The first example comes from Ref. 5, as follows:

$$y'' + y^3 y' - y y' \sqrt{4y' + y^4} = 0 \tag{140}$$

with end conditions

$$(x_1 , y_1) = (0 , 0)$$

$$(x_N , y_N) = (\tan^{-1} 7 , 7)$$

It can be verified that the analytic solution is

$$y = \tan x \tag{141}$$

By Eq. (24), the end conditions are specified by

$$A_1 = 1 , B_1 = 0 , C_1 = 0 , D_1 = 0$$

$$A_N = 1 , B_N = 0 , C_N = 0 , D_N = 7$$

Equations (136) thru (139) yield

$$\alpha(x) = 3 y y' \left(y^2 - \frac{2y' + y^4}{\sqrt{4y' + y^4}} \right) \tag{142}$$

$$\beta(x) = y'\left(\frac{4y' - 3y^4}{\sqrt{4y' + y^4}} - 3y^2\right) \tag{143}$$

$$\gamma(x) = y\left(\frac{6y' + y^4}{\sqrt{4y' + y^4}} - y^2\right) \tag{144}$$

$$\epsilon(x) = 0 \tag{145}$$

The computer program as listed in Appendix A solves the first example problem. The program as listed will be referred to as the reference program.

5.2 EXAMPLE 2: THE FALKNER-SKAN EQUATION

A well-known equation in boundary-layer similarity flows, Ref. 6, is the Falkner-Skan equation,

$$y'' - Iy' + k(1 - y^2) = 0 \tag{146}$$

where k is a parameter. Equation (87) defines I, where

$$I_1 = 0$$

The end conditions are

$$(x_1 , y_1) = (0 , 0)$$

$$(x_N , y_N) = (\infty , 1)$$

For numerical computation, the ∞ must be replaced by an appropriate finite number.

By Eq. (24), the end conditions are specified by

$$A_1 = 1 , B_1 = 0 , C_1 = 0 , D_1 = 0$$

$$A_N = 1 , B_N = 0 , C_N = 0 , D_N = 1$$

Equations (136) thru (139) yield

$$\alpha(x) = Iy' - k(1 + y^2) \tag{147}$$

$$\beta(x) = 2ky \tag{148}$$

$$\gamma(x) = -I \tag{149}$$

$$\epsilon(x) = -y' \tag{150}$$

It took three attempts at solving this problem to obtain satisfactory results in all cases. The goal was to recompute a table in Ref. 6 which is printed in the first page of printout for this example (Appendix E). One could expect difficulty for $k < 0$, since multiple solutions exist, but only the one solution presented in the table was of interest. The first attempt was with $x_N = 6$. Good solutions were obtained for $k \geqslant 0$, but for $k < 0$, the iteration converged to a wrong solution. The second attempt was to choose (using the table) a more appropriate x_N for each k. This improved the accuracy of the solution for $k = 10$, but otherwise it had little effect. The first two attempts used GUESS (Section 4) to obtain the initial solution of the iteration, which produced a straight line. The third attempt was to use as the initial solution

$$y = 1 - e^{-3x} \tag{151}$$

With this as the starting solution, the iteration converged to the correct solution in each case.

The solutions were not printed (LPRNT = 0), but were written on disk (JUNIT = 20). A second program read the file on disk and plotted the solutions and the first derivatives of the solutions. These plots are shown in Figs. 3 and 4 and can be compared with similar plots in Ref. 6.

5.3 EXAMPLE 3: VISCOELASTIC FLUID

The third example comes from Ref. 7 and describes the steady flow of a viscoelastic fluid parallel to an infinite plane surface with uniform suction

$$ky'' + y' + y = 0 \tag{152}$$

with end conditions

$$x_1 = 0 \qquad\qquad x_N = \infty$$

$$I_1 = 0 \qquad\qquad I_N = 1$$

$$y_1 = \lambda$$

where

$$\lambda = \frac{1 - \sqrt{1 - 4k}}{2k} \tag{153}$$

It can be verified that the analytic solution is

$$I = 1 - e^{-\lambda x} \tag{154}$$

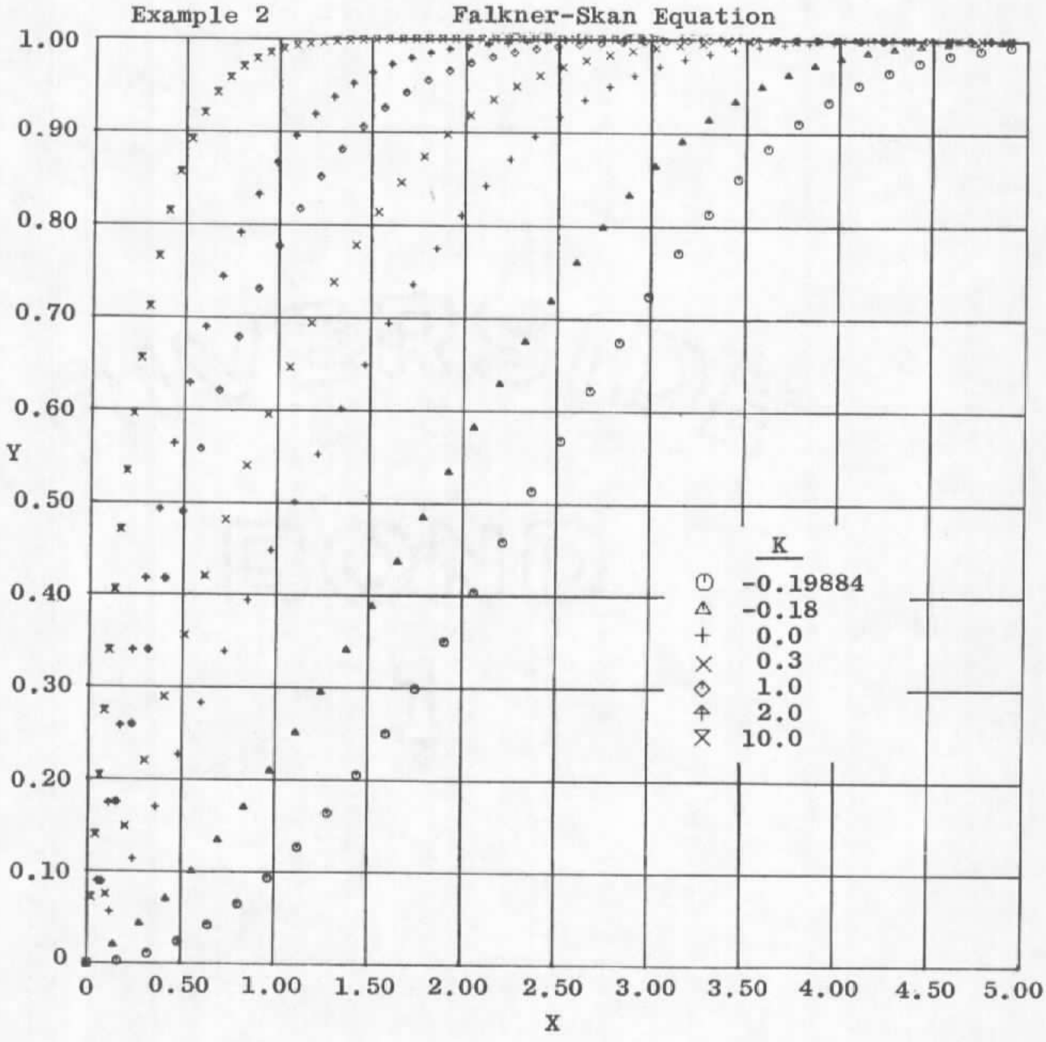

Figure 3. Solutions to the Falkner-Skan equation.

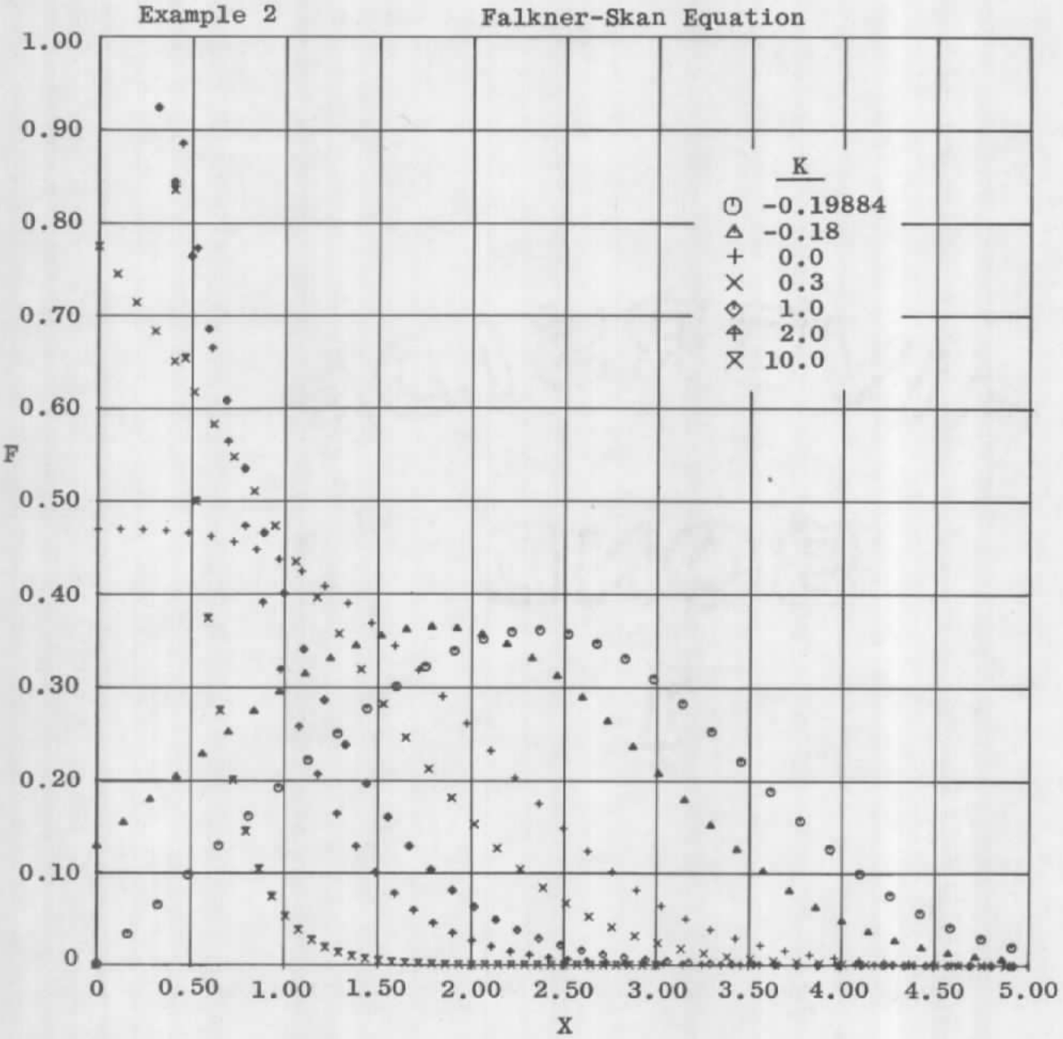

Figure 4. First derivatives of solutions to the Falkner-Skan equation.

The end condition, Eq. (24) with i = 1, is specified by

$$A_1 = 1\ ,\ B_1 = 0\ ,\ C_1 = 0\ ,\ D_1 = \lambda$$

For numerical purposes, x_N must be set to an appropriate finite number. Also, to be noted, the end condition at $x = x_N$ is not of the assumed form, Eq. (24) with i = N. This deviation requires a modification in SYSTEM, replacement of Lines 6750 thru 6780 with

```
D(3,N,4) = 1.D0
```

and replacement of Line 6810 with

```
R(3,N) = 1.D0
```

i.e., replacement of Eq. (24) at i = N with

$$I_N = 1$$

Equation (152) is linear, and from Eq. (92),

$$\alpha = 0 \tag{155}$$

$$\beta = -\frac{1}{k} \tag{156}$$

$$\gamma = -\frac{1}{k} \tag{157}$$

$$\epsilon = 0 \tag{158}$$

Since Eq. (152) is linear, iteration is unnecessary; however, the accuracy is improved by respacing. Solutions were computed for k = 0.001, 0.01, and 0.1. Accuracy improved for larger k values. This was expected since the equation becomes stiff for small k values (Ref. 7). The stiffness is further evidenced by the oscillation of signs in the numerical solution for y'' for the smaller k and larger x values.

5.4 EXAMPLE 4: CHEMICAL DISPERSION

The fourth example,

$$k y'' = y' + \frac{1.25\, y}{1 + 0.1\, y} \tag{159}$$

with end conditions

$$x_1 = 0 \qquad y_1 - k y_1' = 1$$

$$x_N = 1 \qquad y_N' = 0$$

comes from Ref. 7 and is a special case of a chemical dispersion equation. Checkpoints for the numerical solution (taken from Ref. 7) are given below.

	y(x)		
x	k = 1	k = 0.1	k = 0.001
0	0.620042	0.905407	0.999068
0.5	0.473143	0.532646	0.599183
1.0	0.418415	0.339965	0.307476

Using Eq. (24), the end conditions are specified by

$$A_1 = 1 \ , \ B_1 = -k \ , \ C_1 = 0 \ , \ D_1 = 1$$

$$A_N = 0 \ , \ B_N = 1 \ , \ C_N = 0 \ , \ D_N = 0$$

Equations (136) thru (139) yield

$$\alpha(x) = \frac{0.125}{k}\left(\frac{y}{1 + 0.1\,y}\right)^2 \tag{160}$$

$$\beta(x) = \frac{1}{k}\ \frac{1.25}{(1 + 0.1\,y)^2} \tag{161}$$

$$\gamma(x) = \frac{1}{k} \tag{162}$$

$$\epsilon(x) = 0 \tag{163}$$

No difficulties were encountered in obtaining the checkpoint values. A plot of the solutions is shown in Fig. 5.

5.5 EXAMPLE 5: AN INHERENTLY UNSTABLE PROBLEM

The fifth example,

$$y'' = x^2 y + \frac{1}{1 + x} \tag{164}$$

with end conditions

$$(x_1 \ , \ y_1) = (1 \ , \ 1)$$

$$(x_N \ , \ y_N) = (\infty \ , \ 0)$$

comes from Ref. 7, where it is called "an inherently unstable problem."

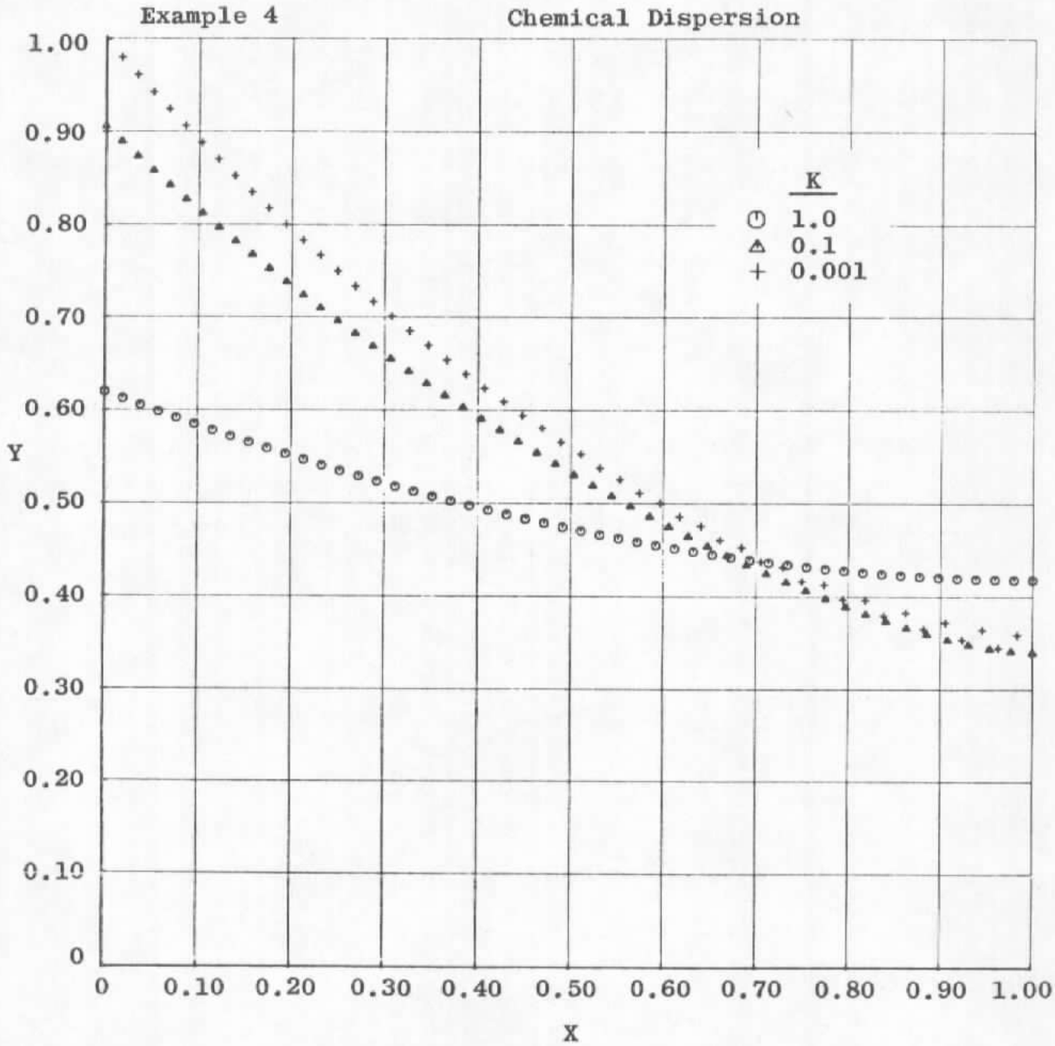

Figure 5. Solutions to the chemical dispersion equation.

By Eq. (24) the end conditions are specified by

$$A_1 = 1\ ,\ B_1 = 0\ ,\ C_1 = 0\ ,\ D_1 = 1$$

$$A_N = 1\ ,\ B_N = 0\ ,\ C_N = 0\ ,\ D_N = 0$$

Equation (164) is linear, and from Eq. (92),

$$\alpha(x) = \frac{1}{1+x} \tag{165}$$

$$\beta(x) = x^2 \tag{166}$$

$$\gamma = 0 \tag{167}$$

and

$$\epsilon = 0 \tag{168}$$

On the first page of printout for this example (Appendix E), the column labeled YA is a tabular solution presented in Ref. 7. The column labeled YB is the computed solution, and the column labeled DIFF is the difference between the two. The accuracy is as good as the accuracy claimed for the table in Ref. 7. A plot of the solution is shown in Fig. 6.

5.6 EXAMPLE 6

The sixth example comes from Ref. 8:

$$y'' = k^2 y + (k^2 + 4\pi^2)\cos^2 \pi x - 2\pi^2 \tag{169}$$

with end conditions

$$(x_1\ ,\ y_1) = (0\ .\ 0)$$

$$(x_N\ ,\ y_N) = (1\ ,\ 0)$$

The analytic solution is

$$y = \frac{e^{k(x-1)} + e^{-kx}}{1 + e^{-k}} - \cos^2 \pi x \tag{170}$$

By Eq. (24) the end conditions are specified by

$$A_1 = 1\ ,\ B_1 = 0\ ,\ C_1 = 0\ ,\ D_1 = 0$$

$$A_N = 1\ ,\ B_N = 0\ ,\ C_N = 0\ ,\ D_N = 0$$

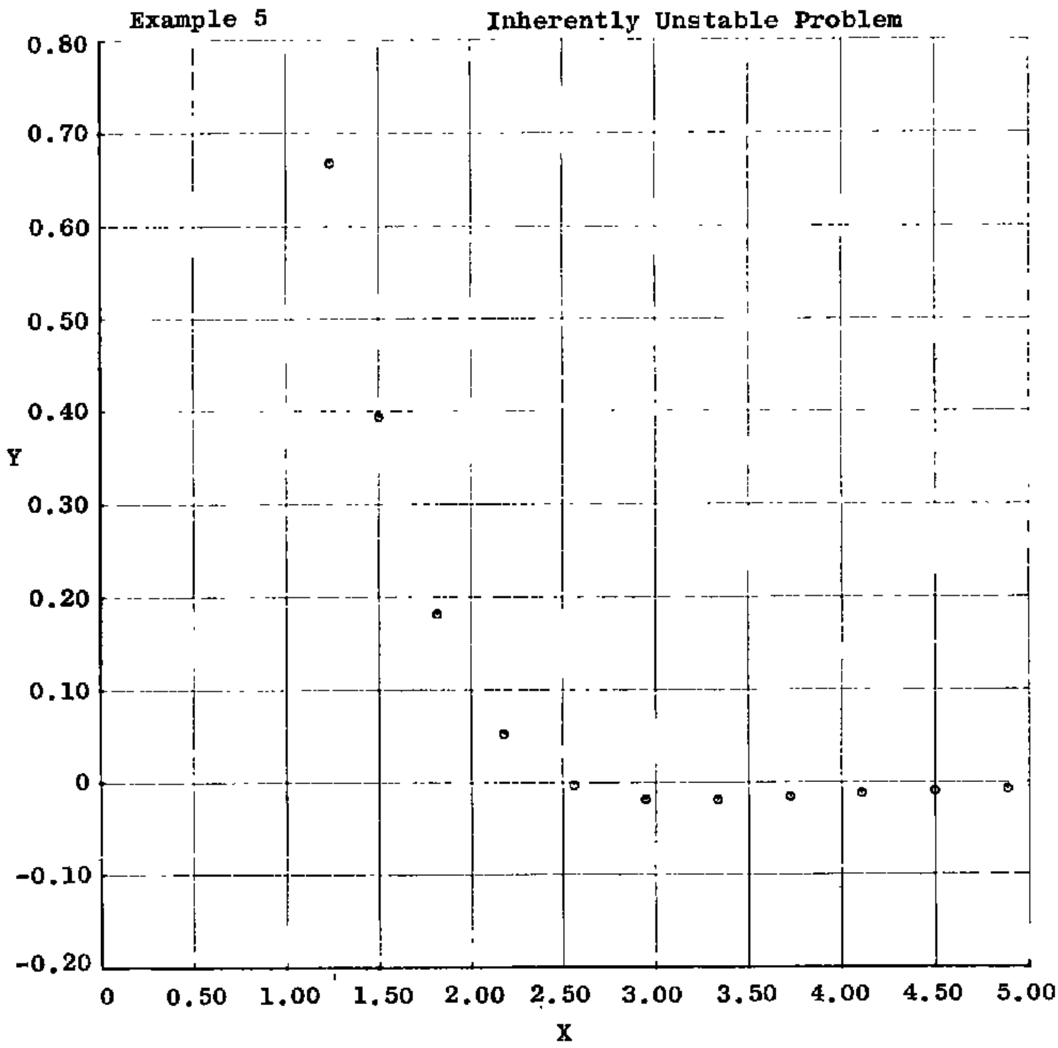

Figure 6. Solution to the inherently unstable problem.

Equation (169) is linear, and by Eq. (92),

$$\alpha(x) = (k^2 + 4\pi^2)\cos^2 \pi x - 2\pi^2 \tag{171}$$

$$\beta = k^2 \tag{172}$$

$$\gamma = 0 \tag{173}$$

$$\epsilon = 0 \tag{174}$$

A plot of solutions to Eq. (169) is shown in Fig. 7.

5.7 EXAMPLE 7: $y'' = e^y$

The seventh example comes from Ref. 8.

$$y'' = e^y \tag{175}$$

with end conditions

$$(x_1, y_1) = (0, 0)$$

$$(x_N, y_N) = (1, 0)$$

The analytic solution is

$$y = \ln\left\{\lambda \sec^2\left[\sqrt{\frac{\lambda}{2}}\left(x - \frac{1}{2}\right)\right]\right\} \tag{176}$$

where λ is the root of

$$\lambda = \cos^2\left(\frac{1}{2}\sqrt{\frac{\lambda}{2}}\right) \tag{177}$$

the numerical value of which can be found in the first page of printout for this example (Appendix E).

Using Eq. (24), the end conditions are specified by

$$A_1 = 1,\ B_1 = 0,\ C_1 = 0,\ D_1 = 0$$

$$A_N = 1,\ B_N = 0,\ C_N = 0,\ D_N = 0$$

Equations (136) thru (139) yield

$$\alpha(x) = (1 - y)\,e^y \tag{178}$$

$$\beta(x) = e^y \tag{179}$$

$$\gamma = 0 \tag{180}$$

$$\epsilon = 0 \tag{181}$$

Of note for this problem was how quickly and accurately the iteration converged to the solution. A plot of the solution is given in Fig. 8.

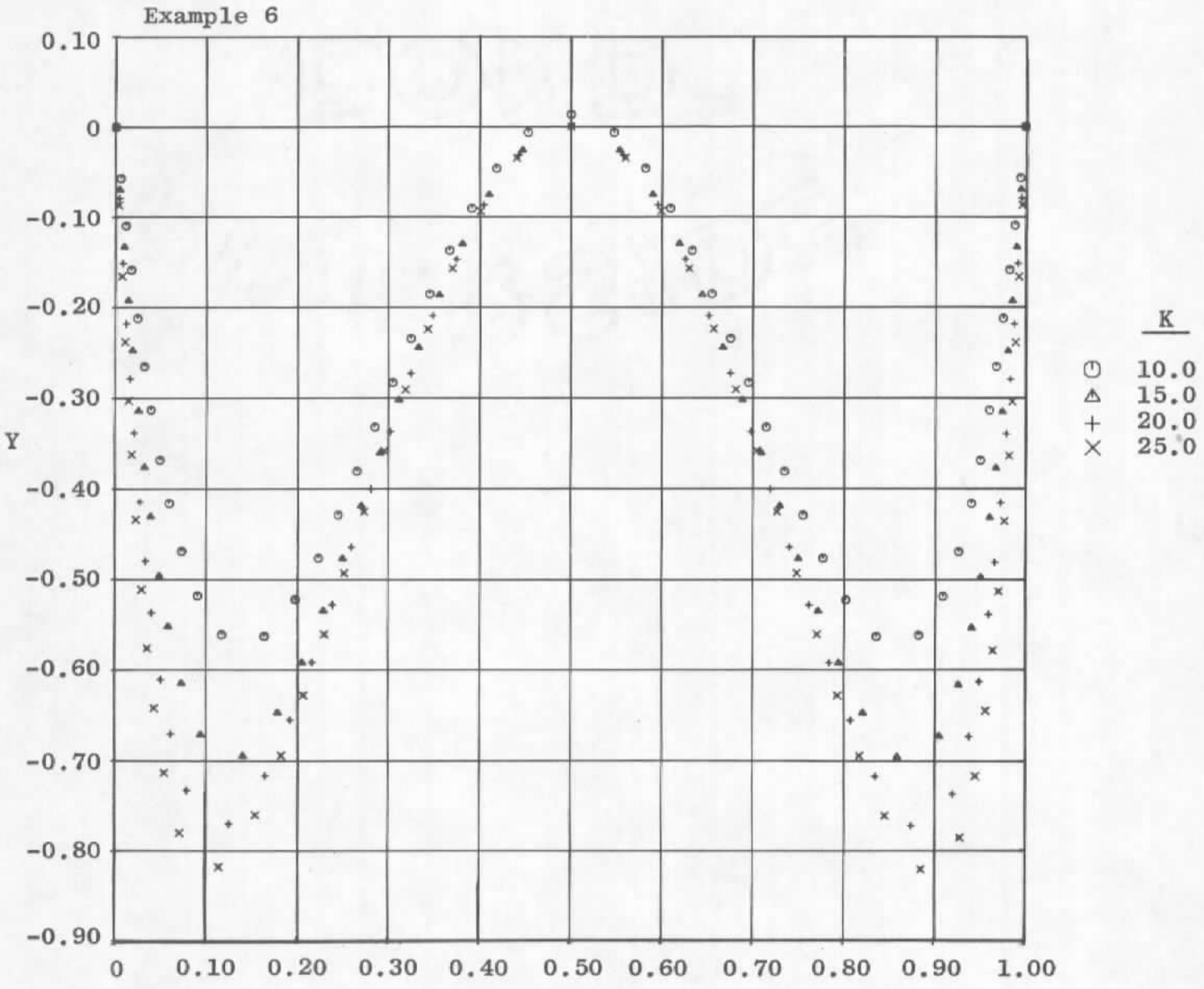

Figure 7. Solutions of example 6.

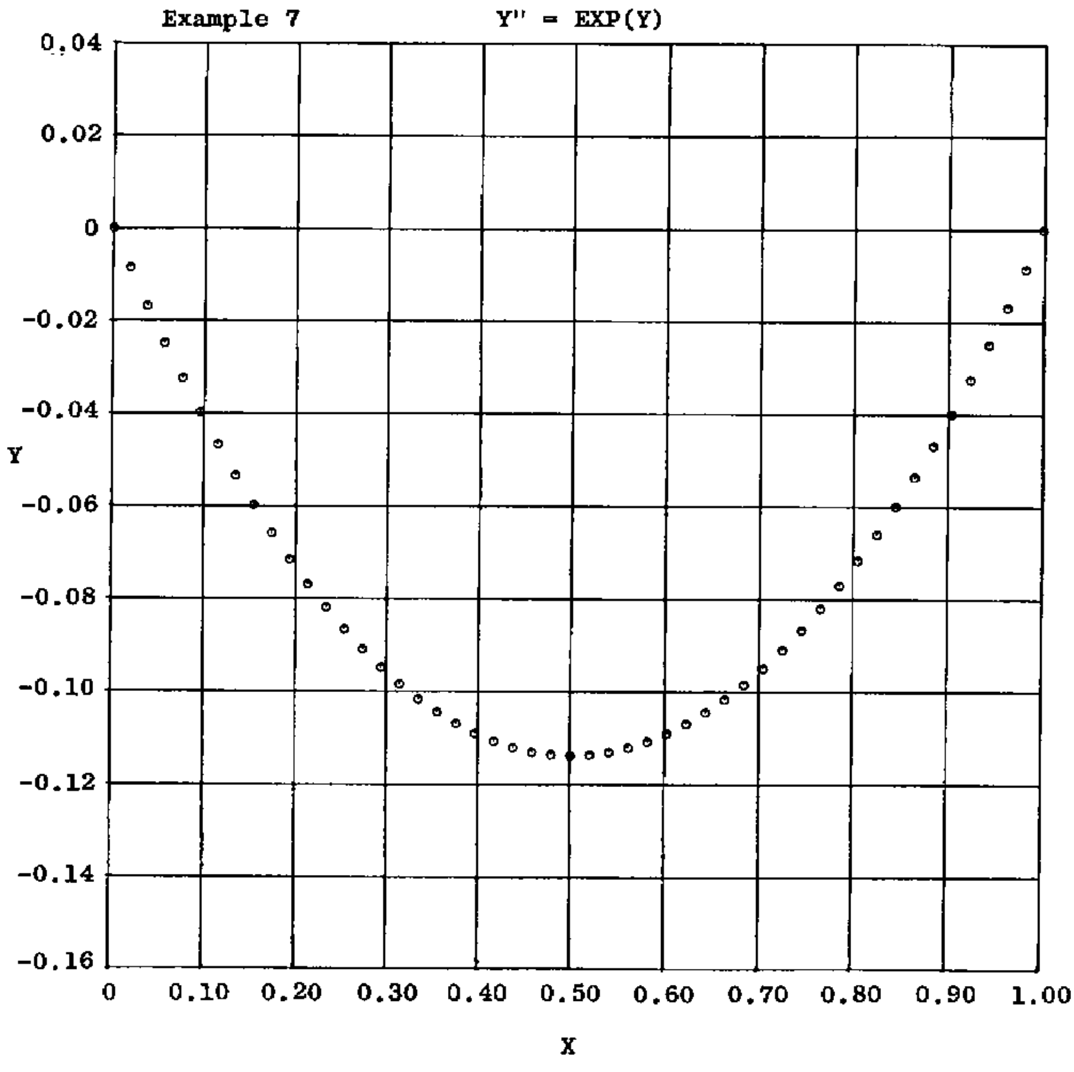

Figure 8. Solution to $y'' = e^y$.

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APPENDIX A
LISTING OF THE COMPUTER PROGRAM

LISTING OF REFERENCE PROGRAM

```
C...  SEVC0628    V32A-P1A    FOR JOHN C. ADAMS
C...  SOLUTION OF 2-POINT BOUNDARY VALUE PROBLEMS BY SPLINE COLLOCATION
C...  WRITTEN BY DON TODD APRIL 7 1978
C...
C...  THE MAIN PROGRAM ALLOCATES MEMORY
C...  ALL REDIMENSIONING CAN BE DONE HERE
C...
      IMPLICIT REAL*8 (A-H,O-Z)
      COMMON /ARRAYS/ A(250,45)
      COMMON /FIXED/ NDIM
      NDIM=250
      CALL PROCED (A(1,1),A(1,2),A(1,3),A(1,4),A(1,5),A(1,6),A(1,7),A(1,
     18),A(1,9),A(1,10),A(1,10),A(1,11),A(1,12),A(1,13))
      WRITE (6,10)
      STOP
10    FORMAT ('0STOP')
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE PROCED (X,Y,F,S,G,ALFA,BETA,GAMA,EPSA,D,XA,YA,FA,GA)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1),ALFA(1),BETA(1),GAMA(1),EPSA(1)
     1,D(1),XA(1),YA(1),FA(1),GA(1)
      COMMON /FIXED/ NDIM,JCASE,JT,NT
C...  PROCEDURE FOR THE
C...  SOLUTION OF 2-POINT BOUNDARY VALUE PROBLEMS BY SPLINE COLLOCATION
C...  WRITTEN BY DON TODD APRIL 7 1978
C...
C...  F IS THE FIRST DERIVATIVE
C...  S IS THE SECOND DERIVATIVE
C...  G IS THE INTEGRAL
C...  D IS THE COEFFICIENT ARRAY FOR THE SYSTEM
C...  XA,YA,FA, & GA ARE THE NEWLY COMPUTED SOLUTION. NOTE THEY OCCUPY
C...  THE SAME MEMORY AS D (SEE CALL PROCED IN THE MAIN PGM)
      CALL PAGES
      DO 50 JCASE=1,100000
      CALL DATA (N,X,Y,F,S,G,&50,&60)
      M=3*N
      JR1=11*M+1
      DO 10 JT=1,NT
      CALL OLDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA)
      CALL SYSTEM (N,X,ALFA,BETA,GAMA,EPSA,D,D(JR1))
      CALL BANDED (11,12,M,6,D,&20)
      CALL RESULT (N,D(JR1),X,YA,FA,S,GA)
      CALL UPDATE (N,Y,F,G,YA,FA,GA,&30)
      CALL RESPAC (N,X,Y,F,S,G,XA,YA,FA,GA)
10    CONTINUE
      WRITE (6,70)
      GO TO 30
20    WRITE (6,80)
      GO TO 50
30    CONTINUE
      CALL OLDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA)
      CALL CHEKDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA)
      CALL CHEKBM (N,X,Y,F,S,G)
      CALL WRITES (N,X,Y,F,S,G)
      CALL PRINTS (N,X,Y,F,S,G)
50    CONTINUE
60    RETURN
70    FORMAT ('0MAXIMUM NUMBER OF ITERATIONS')
80    FORMAT ('0BANDED FAILED')
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE BANDED (NB,N,M,JP,A,*)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION A(M,N)
C...  BANDED SYSTEM SOLVER BY GAUSSIAN ELIMINATION
C...  WRITTEN BY DON TODD APRIL 5 1976
C...
C...  NB IS THE NUMBER OF DIAGONALS (BAND WIDTH)
C...  N IS NUMBER OF EQUATIONS
C...  M IS NB + NUMBER OF RIGHT HAND SIDES (RHS'S)
C...  A IS COEFFICIENT ARRAY AUGMENTED BY RHS'S
C...  SOLUTIONS REPLACE RHS'S   COEFFICIENTS ARE DESTROYED
      NB1=NB+1
      MP1=M+1
      MM1=M-1
      NU=NB-JP
      NL=JP-1
      IF (M.EQ.1) GO TO 300
      IF (NU.EQ.0) GO TO 300
      IF (NL.EQ.0) GO TO 100
      IF (NL.GT.NU) GO TO 200
C...  ELIMINATE LOWER TRIANGLE
      DO 30 J=1,MM1
      P=A(J,JP)
      IF (P.EQ.0.D0) GO TO 400
      Q=1.D0/P
      J1=J+1
      J2=MINO(J+NL,M)
      J3=MINO(J+NU,M)
      DO 30 K=J1,J2
      L1=JP-K+J
      F=A(K,L1)
      IF (F.EQ.0.D0) GO TO 30
      F=F*Q
      L2=JP
      DO 10 L=J1,J3
      L1=L1+1
      L2=L2+1
10    A(K,L1)=A(K,L1)-F*A(J,L2)
      DO 20 L=NB1,N
20    A(K,L)=A(K,L)-F*A(J,L)
30    CONTINUE
C...  BACK SUBSTITUTE
100   CONTINUE
      DO 140 L=NB1,N
      J=MP1
110   J1=J
      J=J-1
      P=A(J,JP)
      IF (P.EQ.0.D0) GO TO 400
      Q=A(J,L)
```

LISTING OF REFERENCE PROGRAM

```
      J2=MIN0(J+NU,M)
      IF (J2.LT.J1) GO TO 130
      L2=JP
      DO 120 K=J1,J2
      L2=L2+1
120   Q=Q-A(J,L2)*A(K,L)
130   A(J,L)=Q/P
      IF (J.GT.1) GO TO 110
140   CONTINUE
      RETURN
C...  ELIMINATE UPPER TRIANGLE
200   J=M
210   P=A(J,JP)
      IF (P.EQ.0.D0) GO TO 400
      Q=1.D0/P
      J1=J-1
      J2=MAX0(J-NU,1)
      J3=MAX0(J-NL,1)
      K=J1
220   L1=JP+J-K
      F=A(K,L1)
      IF (F.EQ.0.D0) GO TO 250
      F=F*Q
      DO 230 L=NB1,N
230   A(K,L)=A(K,L)-F*A(J,L)
      L2=JP
      L=J1
240   L1=L1-1
      L2=L2-1
      A(K,L1)=A(K,L1)-F*A(J,L2)
      IF (L.EQ.J3) GO TO 250
      L=L-1
      GO TO 240
250   IF (K.EQ.J2) GO TO 260
      K=K-1
      GO TO 220
260   IF (J.EQ.2) GO TO 270
      J=J-1
      GO TO 210
270   CONTINUE
C...  FORWARD SUBSTITUTE
300   CONTINUE
      DO 330 L=NB1,N
      DO 330 J=1,M
      P=A(J,JP)
      IF (P.EQ.0.D0) GO TO 400
      Q=A(J,L)
      J2=MAX0(J-NL,1)
      L2=JP
      K=J-1
```

```
310   IF (K.LT.J2) GO TO 320
      L2=L2-1
      Q=Q-A(J,L2)*A(K,L)
      K=K-1
      GO TO 310
320   A(J,L)=Q/P
330   CONTINUE
      RETURN
400   RETURN 1
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE CHEKBM (N,X,Y,F,S,G)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1)
      DIMENSION A(4),E(4),EA(4),KA(4)
C...  USER PROVIDES THIS ROUTINE TO CHECK SOLUTION AS DESIRED.
C...  IF NO CHECKS ARE REQUIRED THEN RETURN
      DO 10 J=1,4
      EA(J)=0.D0
10    KA(J)=0
      DO 30 K=1,N
      SX=1.D0/DCOS(X(K))
      A(1)=DTAN(X(K))
      A(2)=SX**2
      A(3)=2.D0*A(1)*A(2)
      A(4)=DLOG(SX)
      CALL ERROR (A(1),Y(K),E(1))
      CALL ERROR (A(2),F(K),E(2))
      CALL ERROR (A(3),S(K),E(3))
      CALL ERROR (A(4),G(K),E(4))
      DO 20 J=1,4
      IF (E(J).LE.EA(J)) GO TO 20
      EA(J)=E(J)
      KA(J)=K
20    CONTINUE
30    CONTINUE
      WRITE (6,40) KA,EA
      RETURN
40    FORMAT ('0CHEKBM',4I5,1P4E10.2)
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE CHEKDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1),ALFA(1),BETA(1),GAMA(1),EPSA(1
      COMMON /ENDS/G1,A1,B1,C1,D1,AN,BN,CN,DN
C...  CHECK SATISFACTION OF DIFFERENTIAL EQUATION AND END CONDITIONS
C...  WITH SOLUTION
      JM=0
      DM=0.D0
      DO 10 J=1,N
      T=ALFA(J)+BETA(J)*Y(J)+GAMA(J)*F(J)+EPSA(J)*G(J)
      CALL DIFRNC (S(J),T,D)
      IF (D.LE.DM) GO TO 10
      DM=D
      JM=J
10    CONTINUE
      CALL ERROR (G1,G(1),DG)
      T=A1*Y(1)+B1*F(1)+C1*S(1)
      CALL DIFRNC (D1,T,DA)
      T=AN*Y(N)+BN*F(N)+CN*S(N)
      CALL DIFRNC (DN,T,DB)
      WRITE (6,20) JM,DM,DG,DA,DB
      RETURN
20    FORMAT('0CHEKDE',I5,1P4E10.2)
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE DATA (N,X,Y,F,S,G,*,*)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1)
      COMMON /ALPHA/ LAB(18)
      COMMON /ENDS/ G1,A1,B1,C1,D1,AN,BN,CN,DN
      COMMON /FIXED/ NDIM,JCASE,JT,NT,JOUT,LPRNT,LNORM
      COMMON /FLOAT/ TOL,RSC
C...  DATA IS SUPPLIED TO PROGRAM BY USER VIA THIS ROUTINE
      IF (JCASE.GT.1) GO TO 590
C...  PROVIDE ID FOR PROBLEM
      CALL LABELS (LAB    ,'PROBLEM FROM KAMKE                       ')
      CALL LABELS (LAB(10),'                                         ')
C...  N IS NUMBER OF POINTS
C...  NT IS MAXIMUM NUMBER OF ITERATIONS
C...  WILL WRITE SOLUTION ON UNIT JOUT IF JOUT > 0
C...  WILL PRINT SOLUTION IF LPRNT IS NOT ZERO
C...  LNORM CONTROLS WHICH NORM IS USED IN CONVERGENCE TEST (SEE NORM)
C...  ITERATION STOPS IF RELATIVE CHANGE OF SOLUTION IS < TOL
C...  NEW SPACING IS COMPUTED IF CHANGE IN Y IS > RSC
      N=51
      NT=30
      JOUT=0
      LPRNT=1
      LNORM=1
      TOL=5.D-4
      RSC=.01D0
C.... SPECIFY END CONDITIONS
      G1=0.D0
      A1=1.D0
      B1=0.D0
      C1=0.D0
      D1=0.D0
      AN=1.D0
      BN=0.D0
      CN=0.D0
      DN=7.D0
C...  PRINT DATA
      CALL PAGER (100)
      WRITE (6,603)
      WRITE (6,601) NDIM,JCASE,N,NT,JOUT,LPRNT,LNORM
      WRITE (6,604)
      WRITE (6,602) G1,TOL,RSC
      WRITE (6,605)
      WRITE (6,602) A1,B1,C1,D1
      WRITE (6,602) AN,BN,CN,DN
C...  PROVIDE STARTING SOLUTION
      IF (N.GT.NDIM) GO TO 570
      X(1)=0.D0
      X(N)=DATAN(DN)
      CALL UNIFRM (N,X)
```

LISTING OF REFERENCE PROGRAM

```
      CALL GUESS (N,X,Y,F,S,G,&580)
      WRITE (6,606)
      RETURN
C...  TO ABORT A CASE, RETURN 1.
570   WRITE (6,607)
580   RETURN 1
C...  WHEN A JOB IS FINISHED THEN RETURN 2
590   RETURN 2
601   FORMAT (12I10)
602   FORMAT (1P10E12.4)
603   FORMAT (1H0,5X,4HNDIM,5X,5HJCASE,9X,1HN,8X,2HNT,6X,4HJOUT,5X,5HLPR
     1NT,5X,5HLNORM)
604   FORMAT (5H0   I1,10X,3HTOL,9X,3HRSC)
605   FORMAT (4H0   A,11X,1HB,11X,1HC,11X,1HD)
606   FORMAT (1H )
607   FORMAT (9H0N > NDIM)
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE DELTAS (S,DL,DS)
      IMPLICIT REAL*8 (A-H,O-Z)
C...  COMPUTE CAP DELTA, DL, & SMALL DELTA, DS, FROM SIGMA(I-1), S
      DL=S/(S+1.D0)
      DS=(S-1.D0)/(S+1.D0)
      RETURN
      END


      SUBROUTINE DIFRNC (A1,A2,D)
      IMPLICIT REAL*8 (A-H,O-Z)
C...  COMPUTE RELATIVE DIFFERENCE, D, BETWEEN A1 & A2
      D=DABS(A1-A2)
      A=.5D0*DABS(A1+A2)
      IF (A.LE.D) RETURN
      D=D/A
      RETURN
      END


      SUBROUTINE ERROR (A,B,E)
      IMPLICIT REAL*8 (A-H,O-Z)
C...  COMPUTE RELATIVE ERROR, E, OF B FROM EXACT VALUE, A
      E=DABS(A-B)
      D=DABS(A)
      IF (D.LE.E) RETURN
      E=E/D
      RETURN
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE GUESS (N,X,Y,F,S,G,*)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1)
      DIMENSION XA(2),YA(2),FA(2),SA(2)
      COMMON /ENDS/ G1,A1,B1,C1,D1,AN,BN,CN,DN
C...  GUESS SOLUTION IN LACK OF A BETTER APPROXIMATION
C...  WRITTEN BY DON TODD APRIL 13 1978
      H=X(N)-X(1)
      HS=H**2
      BA=B1/H
      BB=BN/H
      CA=C1/HS
      CB=CN/HS
      A11=A1**2+3.D0*(BA**2+CA**2)-2.D0*BA*(A1+CA)
      ANN=AN**2+3.D0*(BB**2+CB**2)+2.D0*BB*(AN+CB)
      A1N=-A1*BB+AN*BA-BA*CB+BB*CA+2.D0*(BA*BB+CA*CB)
      DET=A11*ANN-A1N**2
      IF (DET.EQ.0.D0) GO TO 10
      E1=(ANN*D1-A1N*DN)/DET
      EN=(A11*DN-A1N*D1)/DET
      XA(1)=X(1)
      XA(2)=X(N)
      Q=BA*E1+BB*EN
      YA(1)=A1*E1-Q
      YA(2)=AN*EN+Q
      FT=(YA(2)-YA(1))/H
      Q=CA*E1+CB*EN
      FA(1)=FT+(BA*E1-Q)/H
      FA(2)=FT+(BB*EN+Q)/H
      FT=(FA(2)-FA(1))/H
      SA(1)=FT+CA*E1/HS
      SA(2)=FT+CB*EN/HS
      CALL QUINTS (2,XA,YA,FA,SA,N,X,Y,F,S)
      CALL NTEGRL (N,G1,X,Y,F,S,G)
      WRITE (6,20)
      RETURN
10    WRITE (6,30)
      RETURN 1
20    FORMAT ('0SOLUTION GUESSED')
30    FORMAT ('0CONTRADICTORY END CONDITIONS')
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE NORM (N,Y1,Y2,C,K)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1(1),Y2(1)
      COMMON /FIXED/ NDIM,JCASE,JT,NT,JOUT,LPRNT,LNORM
C...  COMPUTE "DISTANCE" BETWEEN ARRAYS Y1 & Y2
      K=0
      C=0.D0
      IF (LNORM.NE.1) GO TO 30
C...  COMPUTE A GLOBAL NORM
      D=Y1(1)
      R=D
      DO 10 J=1,N
      D=DMIN1(D,Y1(J))
      R=DMAX1(R,Y1(J))
10    CONTINUE
      D=R-D
      IF (D.LE.0.D0) D=R
      IF (D.LE.0.D0) D=1.D0
      DO 20 J=1,N
      R=DABS(Y2(J)-Y1(J))
      IF (R.LE.C) GO TO 20
      C=R
      K=J
20    CONTINUE
      C=C/D
      RETURN
30    IF (LNORM.NE.2) GO TO 50
C...  COMPUTE A LOCAL NORM
      DO 40 J=1,N
      CALL DIFRNC (Y1(J),Y2(J),D)
      IF (D.LE.C) GO TO 40
      C=D
      K=J
40    CONTINUE
50    RETURN
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE NTEGRL (N,G1,X,Y,F,S,G)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1)
C...  COMPUTE INTEGRAL OF QUINTIC SPLINE
      G(1)=G1
      DO 10 J2=2,N
      J1=J2-1
      H=X(J2)-X(J1)
      G(J2)=G(J1)+H*(.5D0*(Y(J1)+Y(J2))+H*(.1D0*(F(J1)-F(J2))
     1 +H*(S(J1)+S(J2))/120.D0))
   10 CONTINUE
      RETURN
      END


      SUBROUTINE PRINTS (N,X,Y,F,S,G)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1)
      COMMON /ALPHA/ LAB(36)
      COMMON /FIXED/ NDIM,JCASE,JT,NT,JOUT,LPRNT
      IF (LPRNT.EQ.0) RETURN
C...  PRINT SOLUTION
      CALL PAGER (100)
      WRITE (6,30)
      HN=X(2)-X(1)
      N2=N-2
      DO 10 J=1,N2
      H=HN
      HN=X(J+2)-X(J+1)
      SIGMA=HN/H
   10 WRITE (6,40) J,X(J),Y(J),F(J),S(J),G(J),H,SIGMA
      J=N-1
      H=HN
      WRITE (6,40) J,X(J),Y(J),F(J),S(J),G(J),H
      J=N
      WRITE (6,40) J,X(J),Y(J),F(J),S(J),G(J)
      RETURN
   30 FORMAT (1H0,10X,1HX,11X,1HY,11X,2HYP,10X,3HYPP,9X,1HI,13X,1HH,10X,
     11HS)
   40 FORMAT (I8,1P5E12.4,E13.3,E10.2)
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE QLDE (N,X,Y,F,S,G,ALFA,BETA,GAMA,EPSA)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1),ALFA(1),BETA(1),GAMA(1),EPSA(1)
C...  USER PROVIDES THIS ROUTINE TO
C...  EVALUATE COEFFICIENTS OF QUASI-LINEARIZED EQUATION
      DO 10 J=1,N
      R=DSQRT(4.D0*F(J)+Y(J)**4)
      ALFA(J)=3.D0*Y(J)*F(J)+(Y(J)**2-(2.D0*F(J)+Y(J)**4)/R)
      BETA(J)=F(J)*((4.D0*F(J)+3.D0*Y(J)**4)/R-3.D0*Y(J)**2)
      GAMA(J)=Y(J)*((6.D0*F(J)+Y(J)**4)/R-Y(J)**2)
      EPSA(J)=0.D0
10    CONTINUE
      RETURN
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE QUINTS (N1,X1,Y1,F1,S1,N2,X2,Y2,F2,S2)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X1(1),Y1(1),F1(1),S1(1),X2(1),Y2(1),F2(1),S2(1)
C...  QUINTIC SPLINE INTERPOLATION
C...  WRITTEN BY DON TODD APRIL 11 1978
      XR=-1.D50
      J2=1
      DO 30 J=1,N2
      IF (X2(J).LE.XR) GO TO 20
10    J1=J2
      J2=J2+1
      XR=X1(J2)
      IF (J2.EQ.N1) XR=1.D50
      IF (X2(J).GT.XR) GO TO 10
      H=X1(J2)-X1(J1)
      YA=Y1(J1)+.2D0*H*F1(J1)
      YB=Y1(J1)+.4D0*H*F1(J1)+.05D0*H**2*S1(J1)
      YC=Y1(J2)-.4D0*H*F1(J2)+.05D0*H**2*S1(J2)
      YD=Y1(J2)-.2D0*H*F1(J2)
20    TA=(X2(J)-X1(J1))/H
      TC=1.D0-TA
      Y2(J)=(Y1(J1)*TC**2+5.D0*YA*TA*TC+10.D0*YB*TA**2)*TC**3
     1      +(Y1(J2)*TA**2+5.D0*YD*TA*TC+10.D0*YC*TC**2)*TA**3
      QA=(TC-4.D0*TA)*TC**3
      QD=(4.D0*TC-TA)*TA**3
      QB=(2.D0*TC-3.D0*TA)*TC**2
      QC=(3.D0*TC-2.D0*TA)*TA**2
      F2(J)=5.D0*(-Y1(J1)*TC**4+QA*YA+2.D0*QB*YB*TA
     1            +Y1(J2)*TA**4+QD*YD+2.D0*QC*YC*TC)/H
      QA=(TC**2-6.D0*TA*TC+3.D0*TA**2)*TC
      QD=(3.D0*TC**2-6.D0*TA*TC+TA**2)*TA
      S2(J)=20.D0*(Y1(J1)*TC**3-QB*YA+QA*YB
     1            +Y1(J2)*TA**3+QC*YD+QD*YC)/H**2
30    CONTINUE
      RETURN
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE RESPAC (N,X,Y,F,S,G,XA,YA,FA,SA)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1),Y(1),F(1),S(1),G(1),XA(1),YA(1),FA(1),SA(1)
      COMMON /ENDS/ G1
      COMMON /FLOAT/ TOL,RSC,DY
C...  DY IS COMPUTED IN UPDATE.  RSC IS SET IN DATA.
      IF (DY.LE.RSC) RETURN
C...  COMPUTE NEW SPACING FOR NEXT ITERATION
C...  WRITTEN BY DON TODD APRIL 13 1978
      N1=N-1
      H=0.D0
      DO 10 J=1,N1
      J1=J+1
10    H=H+DSQRT((X(J1)-X(J))**2+(Y(J1)-Y(J))**2)
      DH=H/N1
      H=DH
      DR=0.D0
      XA(1)=X(1)
      J2=1
      DO 40 J=2,N1
20    IF (DR.GE.H) GO TO 30
      H=H-DR
      XR=X(J2)
      J1=J2
      J2=J2+1
      DX=X(J2)-X(J1)
      DY=Y(J2)-Y(J1)
      DR=DSQRT(DX**2+DY**2)
      CT=DX/DR
      GO TO 20
30    XA(J)=XR+H*CT
      XR=XA(J)
      DR=DR-H
      H=DH
40    CONTINUE
      XA(N)=X(N)
      CALL QUINTS (N,X,Y,F,S,N,XA,YA,FA,SA)
      DO 50 J=2,N1
      X(J)=XA(J)
      Y(J)=YA(J)
      F(J)=FA(J)
50    S(J)=SA(J)
      CALL NTEGRL (N,G1,X,Y,F,S,G)
      WRITE (6,60)
      RETURN
60    FORMAT (' NEW SPACING COMPUTED')
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE RESULT (N,R,X,Y,F,S,G)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION R(3,N),X(1),Y(1),F(1),S(1),G(1)
C...  COMPUTE Y, F, & S FROM RESULT, R, OF BANDED
      HI=X(2)-X(1)
      T1=(R(3,2)-R(3,1))/HI-HI*(2.D0*R(2,1)+R(2,2))/6.D0
      Y(1)=R(3,1)
      F(1)=T1
      S(1)=R(2,1)
      G(1)=R(1,1)
      N1=N-1
      DO 10 I=2,N1
      IA=I+1
      ID=I-1
      HI1=HI
      HI=X(IA)-X(I)
      SI1=HI/HI1
      CALL DELTAS (SI1,DL,DS)
      T1=.5D0*(-R(3,ID)/HI1+(1.D0/HI1-1.D0/HI)*R(3,I)+R(3,IA)/HI)
     1+(HI1*R(2,ID)+2.D0*(HI1-HI)*R(2,I)-HI*R(2,IA))/12.D0
      Q=SI1*R(2,ID)-(SI1+1.D0)*R(2,I)+R(2,IA)
      Y(I)=R(3,I)
      F(I)=T1+DS*HI1*Q/36.D0
      S(I)=R(2,I)+DL*Q/6.D0
      G(I)=R(1,I)
10    CONTINUE
      HI1=HI
      T1=(R(3,N)-R(3,N-1))/HI1+HI1*(2.D0*R(2,N)+R(2,N-1))/6.D0
      Y(N)=R(3,N)
      F(N)=T1
      S(N)=R(2,N)
      G(N)=R(1,N)
      RETURN
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE SYSTEM (N,X,ALFA,BETA,GAMA,EPSA,D,R)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(N),ALFA(N),BETA(N),GAMA(N),EPSA(N),D(3,N, 12),R(3,N)
      COMMON /ENDS/ G1,A1,B1,C1,D1,AN,BN,CN,DN
C...  COMPUTE COEFFICIENTS AND RHS OF SYSTEM
      DO 10 K=1,12
      DO 10 I=1,N
      DO 10 J=1,3
10    D(J,I,K)=0.D0
C...  EQUATIONS FOR I=1
      HI=X(2)-X(1)
      D(1,1,6)=-1.D0
      D(2,1,  5)=EPSA(1)
      D(2,1,  6)=-1.D0-GAMA(1)*HI/3.D0
      D(2,1,  7)=BETA(1)-GAMA(1)/HI
      D(2,1,  9)=-GAMA(1)*HI/6.D0
      D(2,1, 10)=GAMA(1)/HI
      D(3,1,  5)=C1-D1*HI/3.D0
      D(3,1,  6)=A1-B1/HI
      D(3,1,  8)=-B1*HI/6.D0
      D(3,1,  9)=B1/HI
      R(1,1)=-G1
      R(2,1)=-ALFA(1)
      R(3,1)=D1
C...  EQUATIONS FOR I=2
      HI1=HI
      HI=X(3)-X(2)
      SI1=HI/HI1
      CALL DELTAS (SI1,DL,DS)
      AI1=HI1**3*(DL-2.D0*DS)/720.D0
      QC=(GAMA(2)*DS*HI1-6.D0*DL)/36.D0
      D(1,2,  3)=1.D0
      D(1,2,  4)=-HI1**3/30.D0+AI1*SI1
      D(1,2,  5)=9.D0*HI1/20.D0
      D(1,2,6)=-1.D0
      D(1,2,  7)=HI1**3*(2.D0*SI1-3.D0)/120.D0-AI1*(SI1+1.D0)
      D(1,2,  8)=HI1*(1.D0/SI1+11.D0)/20.D0
      D(1,2, 10)=HI1**2*HI/120.D0+AI1
      D(1,2,11)=-HI1/(20.D0*SI1)
      D(2,2,3)=GAMA(2)*HI1/12.D0+SI1*QC
      D(2,2,  4)=-GAMA(2)/(2.D0*HI1)
      D(2,2,  5)=EPSA(2)
      D(2,2,6)=-1.D0+GAMA(2)*(HI1-HI)/6.D0-QC*(SI1+1.D0)
      D(2,2,  7)=BETA(2)+.5D0*GAMA(2)*     (1.D0/HI1-1.D0/HI)
      D(2,2,9)=-GAMA(2)*HI/12.D0+QC
      D(2,2, 10)=GAMA(2)/(2.D0*HI)
      D(3,2,  2)=-HI1/6.D0
      D(3,2,  3)=1.D0/HI1
      D(3,2,  5)=-(HI1+HI)/3.D0
      D(3,2,  6)=-(1.D0/HI1+1.D0/HI)
```

LISTING OF REFERENCE PROGRAM

```
      D(3,2, 8)=-HI/6.D0
      D(3,2,9)=1.D0/HI
      R(1,2)=0.D0
      R(2,2)=-ALFA(2)
      R(3,2)=0.D0
C...  EQUATIONS FOR  2 < I < N
      N1=N-1
      DO 100 I=3,N1
      A2I=HI**3*(DL+2.D0*DS/SI1)/720.D0
      HI2=HI1
      HI1=HI
      HI=X(I+1)-X(I)
      SI2=SI1
      SI1=HI/HI1
      CALL DELTAS (SI1,DL,DS)
      A1I=HI1**3*(DL-2.D0*DS)/720.D0
      QC=(GAMA(I)*DS*HI1-6.D0*DL)/36.D0
      D(1,I, 1)=HI2*HI1**2/120.D0+A2I*SI2
      D(1,I, 2)=-SI2*HI1/20.D0
      D(1,I, 3)=1.D0
      D(1,I, 4)=HI1**3*(1.D0/SI2-1.D0)/60.D0+A1I*SI1-A2I*(SI2+1.D0)
      D(1,I, 5)=HI1*(SI2+10.D0)/20.D0
      D(1,I, 6)=-1.D0
      D(1,I, 7)=HI1**3*(SI1-1.D0)/60.D0-A1I*(SI1+1.D0)+A2I
      D(1,I, 8)=HI1*(1.D0/SI1+10.D0)/20.D0
      D(1,I,10)=HI1**2*HI/120.D0+A1I
      D(1,I, 11)=-HI1/(20.D0*SI1)
      D(2,I,3)=GAMA(I)*HI1/12.D0+SI1*QC
      D(2,I, 4)=-GAMA(I)/(2.D0*HI1)
      D(2,I, 5)=EPSA(I)
      D(2,I,6)=-1.D0+GAMA(I)*(HI1-HI)/6.D0-QC*(SI1+1.D0)
      D(2,I, 7)=BETA(I)+.5D0*GAMA(I)*(1.D0/HI1-1.D0/HI)
      D(2,I,9)=-GAMA(I)*HI/12.D0+QC
      D(2,I, 10)=GAMA(I)/(2.D0*HI)
      D(3,I, 2)=-HI1/6.D0
      D(3,I, 3)=1.D0/HI1
      D(3,I, 5)=-(HI1+HI)/3.D0
      D(3,I, 6)=-(1.D0/HI1+1.D0/HI)
      D(3,I, 8)=-HI/6.D0
      D(3,I, 9)=1.D0/HI
      R(1,I)=0.D0
      R(2,I)=-ALFA(I)
      R(3,I)=0.D0
100   CONTINUE
C...  EQUATIONS FOR  I=N
      A2I=HI**3*(DL+2.D0*DS/SI1)/720.D0
      HI2=HI1
      HI1=HI
      SI2=SI1
      D(1,N, 1)=HI2*HI1**2/120.D0+A2I*SI2
```

LISTING OF REFERENCE PROGRAM

```
      D(1,N,  2)=-SI2*HI1/20.D0
      D(1,N,  3)=1.D0
      D(1,N,  4)=HI1**3*(2.D0/SI2-3.D0)/120.D0-A2I*(SI2+1.D0)
      D(1,N,  5)=HI1*(SI2+11.D0)/20.D0
      D(1,N,6)=-1.D0
      D(1,N,  7)=-HI1**3/30.D0+A2I
      D(1,N,  8)=9.D0*HI1/20.D0
      D(2,N,  3)=GAMA(N)*HI1/6.D0
      D(2,N,  4)=-GAMA(N)/HI1
      D(2,N,  5)=EPSA(N)
      D(2,N,  6)=-1.D0+GAMA(N)*HI1/3.D0
      D(2,N,  7)=BETA(N)+GAMA(N)/HI1
      D(3,N,  2)=BN*HI1/6.D0
      D(3,N,  3)=-BN/HI1
      D(3,N,  5)=CN+BN*HI1/3.D0
      D(3,N,  6)=AN+BN/HI1
      R(1,N)=0.D0
      R(2,N)=-ALFA(N)
      R(3,N)=DN
      RETURN
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE UNIFRM (N,X)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(1)
C...  GIVEN N, X(1), & X(N) COMPUTE UNIFORM SPACING
      N1=N-1
      DX=(X(N)-X(1))/N1
      DO 10 J=2,N1
10    X(J)=X(1)+(J-1)*DX
      WRITE (6,20)
      RETURN
20    FORMAT ('0UNIFORM SPACING COMPUTED')
      END


      SUBROUTINE UPDATE (N,Y1,F1,G1,Y2,F2,G2,*)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION Y1(1),F1(1),G1(1),Y2(1),F2(1),G2(1)
      DIMENSION KA(3)
      COMMON /FIXED/ NDIM,JCASE,JT
      COMMON /FLOAT/ TOL,RSC,E(3)
C...  COMPUTE CHANGE IN SOLUTION
      CALL NORM (N,Y1,Y2,E(1),KA(1))
      CALL NORM (N,F1,F2,E(2),KA(2))
      CALL NORM (N,G1,G2,E(3),KA(3))
      WRITE (6,6001) JT,KA,E
C...  UPDATE SOLUTION
      DO 50 K=1,N
      Y1(K)=Y2(K)
      F1(K)=F2(K)
50    G1(K)=G2(K)
C...  CHECK CONVERGENCE
      DO 60 J=1,3
      IF (E(J).GT.TOL) GO TO 70
60    CONTINUE
C...  IF CONVERGED THEN RETURN 1
      WRITE (6,6002)
      RETURN 1
70    RETURN
6001  FORMAT (' UPDATE',4I10,1P3E10.2)
6002  FORMAT ('0CONVERGED SOLUTION')
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE WRITES (N,X,Y,F,S,G)
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(N),Y(N),F(N),S(N),G(N)
      COMMON /ALPHA/ LAB(36)
      COMMON /FIXED/ NDIM,JCASE,JT,NT,JOUT
      IF (JOUT.LE.0) RETURN
C...  WRITE SOLUTION ON UNIT JOUT
      WRITE (JOUT) LAB,N
      WRITE (JOUT) X,Y,F,S,G
      WRITE (6,50) JOUT
      RETURN
50    FORMAT ('0SOLUTION WRITTEN ON UNIT',I3)
      END
```

```
      SUBROUTINE PAGER (N)
      COMMON /ALPHA/ LAB(18),ID1(9),ID2(9),JLINE,NLINE,JPAGE
C...  LINE COUNTER ROUTINE
      JLINE=JLINE+N
      IF (JLINE.LE.NLINE) RETURN
      JLINE=N
      JPAGE=JPAGE+1
      CALL TIMCHK (T)
      M=T/60.
      S=T-60*M
      T=M+S/100.
      WRITE (6,10) ID1,ID2,JPAGE
      WRITE (6,20) LAB,T
      RETURN
      ENTRY PAGES
C...  INITIALIZE LINE COUNTER
      CALL TIMCHK (T)
      CALL LABELS (LAB,36H                                    )
      CALL LABELS (LAB(10),LAB)
C...  FOLLOWING CARD SUPPLIES USER ID
      CALL LABELS (ID1,36HDC TODD  CDPS                       )
      CALL GETNOW (ID2)
      JPAGE=0
      NLINE=50
      JLINE=NLINE+1
      RETURN
10    FORMAT (1H1,18A4,38X,4HPAGE,I6)
20    FORMAT (1H ,18A4,38X,4HTIME,F6.2)
      END
```

LISTING OF REFERENCE PROGRAM

```
      SUBROUTINE LABELS (L,N)
      DIMENSION L(1), N(1)
C...  TRANSFER LITERAL INTO AN ARRAY
      DO 10 J=1,9
10    L(J)=N(J)
      RETURN
      END


      SUBROUTINE TIMCHK (T)
C...  SYSTEM DEPENDENT ROUTINE TO PROVIDE USED CPU TIME IN SECONDS
      T=0.
      RETURN
      END


      SUBROUTINE GETNOW (N)
      DIMENSION N(1)
C...  SYSTEM DEPENDENT ROUTINE TO PROVIDE SHOT ID IN FOLLOWING FORM
      CALL LABELS (N,'AR008671    10:29   WED MAY 10, 1978')
      RETURN
      END
```

APPENDIX B
INITIAL SOLUTION

To begin iteration (Section 3.2) an initial "in-hand" solution is needed, and at times it cannot be justified as anything more than a guess. Even a guess should be made as consistent as possible, however. In particular, it should satisfy the end conditions, Eq. (24). Since the spline is a quintic in each interval, $[x_i, x_{i+1}]$, it seems reasonable that a quintic over the whole interval, $[x_1, x_N]$, could possibly be a good guess. Equation (24) with i = 1 and i = N gives two equations in six unknowns. If further assumptions are made to fully specify the six unknowns, then the quintic would be determined. The following procedure is proposed to specify a qunitic with the least variation in itself and its derivative.

Define

$$H = x_N - x_1 \tag{B-1}$$

$$\bar{m} = \frac{1}{H}(y_N - y_1) \tag{B-2}$$

$$\bar{M} = \frac{1}{H}(m_N - m_1) \tag{B-3}$$

$$E = y_1^2 + y_N^2 + H^2\left[(m_1 - \bar{m})^2 + (m_N - \bar{m})^2\right]$$

$$+ H^4\left[(M_1 - \bar{M})^2 + (M_N - \bar{M})^2\right] \tag{B-4}$$

It is proposed that E be minimized subject to the constraints of Eq. (24) with i = 1 and i = N. Using the Lagrange method of multipliers to minimize E, one defines

$$F = E - 2\lambda_1(A_1 y_1 + B_1 m_1 + C_1 M_1 - D_1)$$

$$- 2\lambda_N(A_N y_N + B_N m_N + C_N M_N - D_N) \tag{B-5}$$

which introduces two extra unknowns, λ_1 and λ_N. Differentiating yields

$$\frac{\partial F}{\partial y_1} = 2y_1 + 2H(m_1 + m_N - 2\bar{m}) - 2A_1\lambda_1 = 0 \tag{B-6}$$

$$\frac{\partial F}{\partial y_N} = 2y_N - 2H(m_1 + m_N - 2\bar{m}) - 2A_N\lambda_N = 0 \tag{B-7}$$

$$\frac{\partial F}{\partial m_1} = 2H^2(m_1 - \bar{m}) + 2H^3(M_1 + M_N - 2\bar{M}) - 2B_1\lambda_1 = 0 \tag{B-8}$$

$$\frac{\partial F}{\partial m_N} = 2H^2(m_N - \bar{m}) - 2H^3(M_1 + M_N - 2\bar{M}) - 2B_N\lambda_N = 0 \tag{B-9}$$

$$\frac{\partial F}{\partial M_1} = 2H^4 (M_1 - \tilde{M}) - 2C_1 \lambda_1 = 0 \tag{B-10}$$

$$\frac{\partial F}{\partial M_N} = 2H^4 (M_N - \tilde{M}) - 2C_N \lambda_N = 0 \tag{B-11}$$

$$\frac{\partial F}{\partial \lambda_1} = -2(A_1 y_1 + B_1 m_1 + C_1 M_1 - D_1) = 0 \tag{B-12}$$

$$\frac{\partial F}{\partial \lambda_N} = -2(A_N y_N + B_N m_N + C_N M_N - D_N) = 0 \tag{B-13}$$

The last two equations are the equations of constraint. Equations (B-6) thru (B-13), after substitution of Eqs. (B-2) and (B-3) for $\tilde{m}$ and $\tilde{M}$, are eight linear equations in eight unknowns. One can simplify the system as follows. Adding Eqs. (B-8) and (B-9), one obtains

$$H(m_1 + m_N - 2\tilde{m}) = \frac{1}{H}(B_1 \lambda_1 + B_N \lambda_N) \tag{B-14}$$

Substituting Eq. (B-14) into Eqs. (B-6) and (B-7)

$$y_1 = A_1 \lambda_1 - \frac{1}{H}(B_1 \lambda_1 + B_N \lambda_N) \tag{B-15}$$

$$y_N = A_N \lambda_N + \frac{1}{H}(B_1 \lambda_1 + B_N \lambda_N) \tag{B-16}$$

Substituting Eqs. (B-15) and (B-16) into Eq. (B-2),

$$\tilde{m} = \frac{1}{H}(A_N \lambda_N - A_1 \lambda_1) + \frac{2}{H^2}(B_1 \lambda_1 + B_N \lambda_N) \tag{B-17}$$

Adding Eqs. (B-10) and (B-11),

$$H^2 (M_1 + M_N - 2\tilde{M}) = \frac{1}{H^2}(C_1 \lambda_1 + C_N \lambda_N) \tag{B-18}$$

Substituting Eqs. (B-17) and (B-18) into Eqs. (B-8) and (B-9),

$$\begin{aligned} m_1 = {} & \frac{1}{H}(A_N \lambda_N - A_1 \lambda_1) + \frac{2}{H^2}(B_1 \lambda_1 + B_N \lambda_N) \\ & - \frac{1}{H^3}(C_1 \lambda_1 + C_N \lambda_N) + \frac{1}{H^2} B_1 \lambda_1 \end{aligned} \tag{B-19}$$

$$\begin{aligned} m_N = {} & \frac{1}{H}(A_N \lambda_N - A_1 \lambda_1) + \frac{2}{H^2}(B_1 \lambda_1 + B_N \lambda_N) \\ & + \frac{1}{H^3}(C_1 \lambda_1 + C_N \lambda_N) + \frac{1}{H^2} B_N \lambda_N \end{aligned} \tag{B-20}$$

Substituting Eqs. (B-19) and (B-20) into Eq. (B-3),

$$\tilde{M} = \frac{2}{H^4}(C_1\lambda_1 + C_N\lambda_N) + \frac{1}{H^3}(B_N\lambda_N - B_1\lambda_1) \tag{B-21}$$

Substituting Eq. (B-21) into Eqs. (B-10) and (B-11),

$$M_1 = \frac{2}{H^4}(C_1\lambda_1 + C_N\lambda_N) + \frac{1}{H^3}(B_N\lambda_N - B_1\lambda_1) + \frac{1}{H^4}C_1\lambda_1 \tag{B-22}$$

$$M_N = \frac{2}{H^4}(C_1\lambda_1 + C_N\lambda_N) + \frac{1}{H^3}(B_N\lambda_N - B_1\lambda_1) + \frac{1}{H^4}C_N\lambda_N \tag{B-23}$$

If Eqs. (B-15), (B-16), (B-19), (B-20), (B-22), and (B-23) are substituted into Eqs. (B-12) and (B-13), then they take the form

$$A_{11}\lambda_1 + A_{1N}\lambda_N = D_1 \tag{B-24}$$

$$A_{N1}\lambda_1 + A_{NN}\lambda_N = D_N \tag{B-25}$$

where

$$A_{11} = A_1^2 + 3\overline{B}_1^2 + 3\overline{C}_1^2 - 2A_1\overline{B}_1 - 2\overline{B}_1\overline{C}_1 \tag{B-26}$$

$$A_{1N} = -A_1\overline{B}_N + A_N\overline{B}_1 - \overline{B}_1\overline{C}_N + \overline{B}_N\overline{C}_1 + 2\overline{B}_1\overline{B}_N + 2\overline{C}_1\overline{C}_N \tag{B-27}$$

$$A_{N1} = A_{1N} \tag{B-28}$$

$$A_{NN} = A_N^2 + 3\overline{B}_N^2 + 3\overline{C}_N^2 + 2A_N\overline{B}_N + 2\overline{B}_N\overline{C}_N \tag{B-29}$$

$$\overline{B}_i = \frac{1}{H}B_i \tag{B-30}$$

$$\overline{C}_i = \frac{1}{H^2}C_i \tag{B-31}$$

with $i = 1$ and $i = N$. Equations (B-24) and (B-25) can be solved for λ_1 and λ_N. One can then obtain y_1, y_N, m_1, m_N, M_1, and M_N from Eqs. (B-15), (B-16), (B-19), (B-20), (B-22), and (B-23). These values determine the quintic which can be written down directly by analogy with Eq. (86). This is the initial solution suggested, for lack of a better approximation.

APPENDIX C
CHANGING SPACING

Splines were originally invented for interpolation, and interpolation of spline solutions is immediately available. This, plus the suitability of spline collocation to handle nonuniform spacing, makes it feasible to change the spacing between iterations to improve accuracy.

Since smaller steps are needed where the function changes the fastest, the following procedure can be used. Consider straight line segments between points, (x_i, y_i) (that is, a broken-line solution). Compute the length of the broken line and divide by (N-1) to determine a constant step size along its length. Taking constant steps along the broken-line solution, determine the new x_i. Interpolation provides the new in-hand solution for the next iteration.

Figure C-1 illustrates the respacing technique with N = 6. Beginning with equal spacing, the old solution is marked by the heavy dots. The new spacing is determined by taking equal steps along the broken line, marked by the symbol "Δ". The new solution is obtained by interpolation at the new x_i and is marked by the symbol "□". Note the shorter h_i at the ends, where the function is changing the fastest, and the longer h_i in the middle, where the function is changing the slowest.

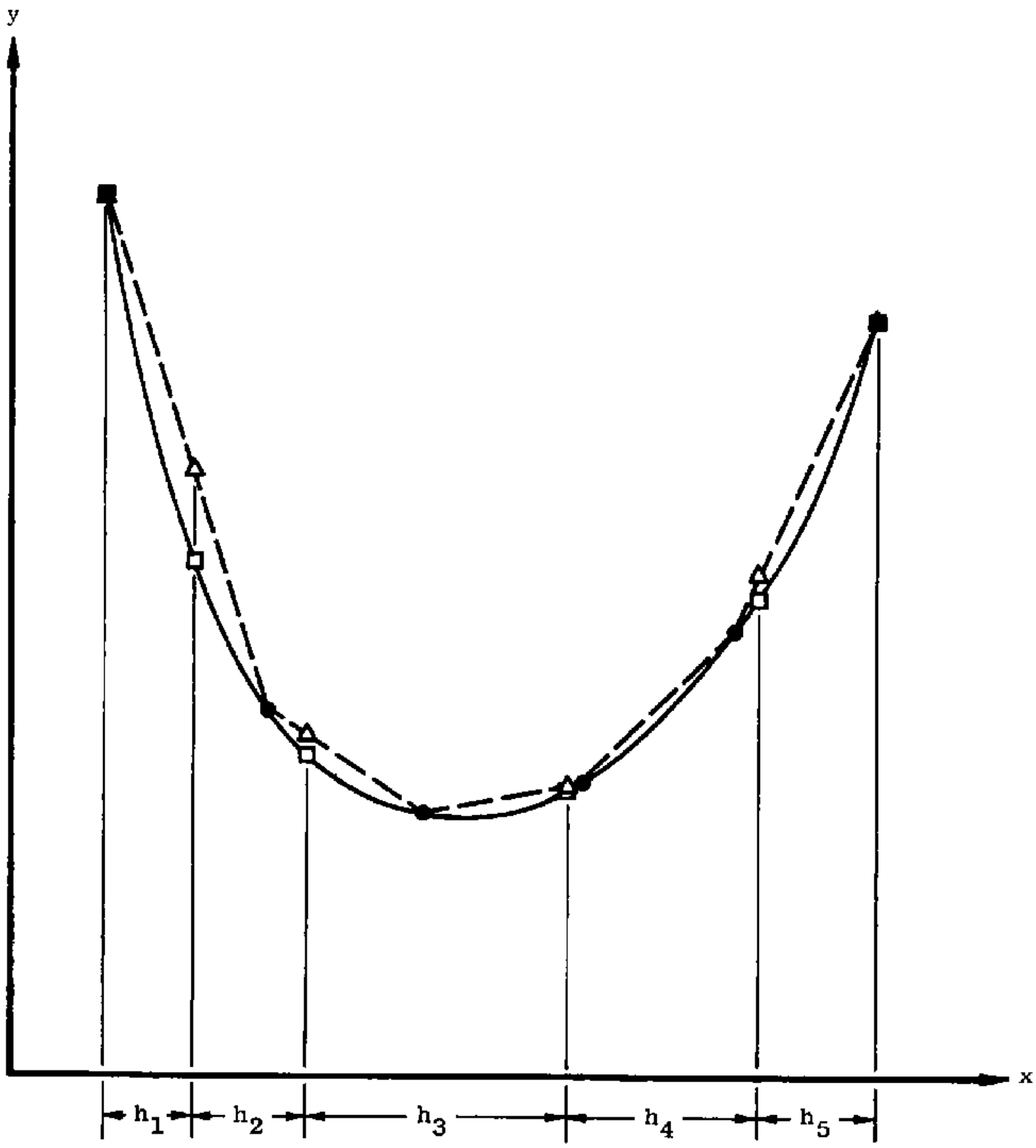

Figure C-1. Respacing technique.

APPENDIX D
MODIFICATIONS TO RUN THE EXAMPLE PROBLEMS

MODIFICATIONS TO RUN EXAMPLE 2

```
-INS  1740
      DIMENSION YB(28),FB(28),SB(28)
      COMMON /ENDS/ G1
      COMMON /FIXED/ NDIM,JCASE
      COMMON /PARM/ PA(7),FA(7),XA(28),YA(7,28)
-REP  1800,1900
      M=15
      DO 14 K=16,28
      IF (YA(JCASE,K).EQ.1.D0) GO TO 16
14    N=M+1
      M=28
16    CALL QUINTS (N,X,Y,F,S,M,XA,YB,FB,SB)
      CALL ERROR (FA(JCASE),F(1),EA(2))
      DO 30 K=1,M
      CALL ERROR (YA(JCASE,K),YB(K),E(1))
      DO 20 J=1,1
-REP  2310,2350
      COMMON /PARM/ PA(7),FA(7),XA(28),YA(7,28)
      DIMENSION XN(7)
      DATA XN/8.D0,7.D0,6.5D0,6.D0,5.5D0,4.8D0,3.D0/
      IF (JCASE.GT.7) GO TO 590
      IF (JCASE.GT.1) GO TO 30
      CALL LABELS (LAB,36HEXAMPLE 2
      CALL LABELS (LAB(10),36HFALKNER-SKAN EQUATION
      READ (5,5001) PL,PA
      READ (5,5001) FL,FA
      DO 10 K=1,28
10    READ (5,5003) XA(K),(YA(J,K),J=1,7)
      DO 15 K=16,28
      DO 15 J=5,7
      IF (YA(J,K).EQ.0.D0) YA(J,K)=1.D0
15    CONTINUE
      CALL PAGER (100)
      WRITE (6,6001) PL,PA
      WRITE (6,6001) FL,FA
      WRITE (6,6002)
      DO 20 K=1,28
20    WRITE (6,6003) K,XA(K),(YA(J,K),J=1,7)
30    CONTINUE
-REP  2450,2460
      JOUT=20
      LPRNT=0
-REP  2590
      DN=1.D0
-INS  2610
      WRITE (6,6004) PA(JCASE)
-REP  2720
      X(N)=XN(JCASE)
-REP  2740
      A=3.D0
```

MODIFICATIONS TO RUN EXAMPLE 2

```
      DO 50 J=1,N
      EX=DEXP(-A*X(J))
      Y(J)=1.D0-EX
      F(J)=A*EX
      S(J)=-EX*A**2
      G(J)=X(J)+(EX-1.D0)/A
50    CONTINUE
-INS  2890
5001  FORMAT (A4,5X,7E9.0)
5003  FORMAT (8E9.0)
6001  FORMAT (1H0,A4,5X,7F10.5)
6002  FORMAT (1H0,7X,42HX      Y(X) CORRESPONDING TO ABOVE K VALUES)
6003  FORMAT (I3,F7.1,7F10.5)
6004  FORMAT ('0K =',F10.5)
-INS  4340
      COMMON /FIXED/ NDIM,JCASE
      COMMON /PARM/ PA(7),FA(7),XA(28),YA(7,28)
-INS  4360
      P=PA(JCASE)
-REP  4380,4420
      ALFA(J)=G(J)*F(J)-P*(1.D0+Y(J)**2)
      BETA(J)=2.D0*P*Y(J)
      GAMA(J)=-G(J)
      EPSA(J)=-F(J)
```

MODIFICATIONS TO RUN EXAMPLE 3

```
-INS   1740
       COMMON /PARM/ P,Q
-REP   1810,1890
       Z=DEXP(-Q*X(K))
       A(1)=Q*Z
       A(2)=-Q*A(1)
       A(3)=-Q*A(2)
       A(4)=1.D0-Z
       E(1)=DABS(A(1)-Y(K))
       E(2)=DABS(A(2)-F(K))
       E(3)=DABS(A(3)-S(K))
       E(4)=DABS(A(4)-G(K))
-INS   2300
       COMMON /PARM/ P,Q
-REP   2320
       IF (JCASE.EQ.1) GO TO 20
       IF (JCASE.EQ.2) GO TO 10
       IF (JCASE.GT.3) GO TO 590
       P=.1D0
       GO TO 30
10     P=.01D0
       GO TO 30
20     P=.001D0
30     Q=( 1.D0-DSQRT(1.D0-4.D0*P))/(2.D0*P)
-REP   2340,2350
       CALL LABELS (LAB,36HEXAMPLE 3
       CALL LABELS (LAB(10),36HVISCOELASTIC FLUID
-REP   2550
       D1=Q
-INS   2610
       WRITE (6,700) P,Q
-REP   2720
       X(N)=12.D0/Q
-REP   2740
       DO 40 J=1,N
       Y(J)=1.D0
       F(J)=1.D0
       S(J)=1.D0
40     G(J)=1.D0
-INS   2890
700    FORMAT (4H0K =,1PE12.4,5X,8HLAMBDA =,E12.4)
-INS   4340
       COMMON /PARM/ P,Q
-REP   4380,4410
       ALFA(J)=0.D0
       BETA(J)=-1.D0/P
       GAMA(J)=BETA(J)
-REP   6750,6780
       D(3,N,4)=1.D0
-REP   6810
       R(3,N)=1.D0
```

MODIFICATIONS TO RUN EXAMPLE 4

```
-REP   1740
       DIMENSION XA(3),YA(3,3),YB(3),FB(3),SB(3),EA(4),KA(4)
       COMMON /FIXED/ NDIM,JCASE
       DATA XA/0.D0,.5D0,1.D0/
       DATA YA/.62004200,.47314330,.41841500,.90540700,.53264600,.3399650
      1D,.99906800,.55918300,.30747600/
-REP   1800,1950
       CALL QUINTS (N,X,Y,F,S,3,XA,YB,FB,SB)
       DO 20 J=1,3
20     CALL ERROR (YA(J,JCASE),YB(J),EA(J))
-INS   2300
       COMMON /PARM/ P
       DIMENSION PA(3)
       DATA PA/1.D0,.1D0,.001D0/
-REP   2320
       IF (JCASE.GT.3) GO TO 590
       P=PA(JCASE)
-REP   2340,2350
       CALL LABELS (LAB,36HEXAMPLE 4
       CALL LABELS (LAB(10),36HCHEMICAL DISPERSION
-REP   2450,2460
       JOUT=20
       LPRNT=1
-REP   2530,2590
       B1=-P
       C1=0.D0
       D1=1.D0
       AN=0.D0
       BN=1.D0
       CN=0.D0
       DN=0.D0
-INS   2610
       WRITE (6,6004) P
-REP   2720
       X(N)=1.D0
-INS   2890
6004   FORMAT ('OK =',F10.5)
-INS   4340
       COMMON /PARM/ P
-REP   4380,4410
       BETA(J)=1.25D0/(P*(1.D0+.1D0*Y(J))**2)
       ALFA(J)=.1D0*BETA(J)*Y(J)**2
       GAMA(J)=1.D0/P
```

MODIFICATIONS TO RUN EXAMPLE 5

```
-REP   1740
       DIMENSION XA(20),YA(20),YB(20),DM(20)
       DATA XA/1.D0,1.2D0,1.4D0,1.6D0,1.8D0,2.0D0,2.2D0,2.4D0,2.6D0,2.8D0
      1,3.0D0,3.2D0,3.4D0,4.D0,5.D0/
       DATA YA/1.D0,.714D0,.4971D0,.3146D0,.1897D0,.1039D0,.0475D0,.0134D
      10,-.0055D0,-.0155D0,-.0192D0,-.0192D0,-.0181D0,-.0131D0,-.0068D0/
-REP   1770,1980
       CALL QUINTS (N,X,Y,F,5,15,XA,YB,DM,DM)
       WRITE (6,400)
       DO 50 J=1,15
       D=YA(J)-YB(J)
       WRITE (6,410) J,XA(J),YA(J),YB(J),D
50     CONTINUE
       RETURN
400    FORMAT ('0COMPARISON OF SOLUTION WITH TABLE IN REF 7'/1H0,8X,1HJ,8
      1X,1HX,6X,2HYA,8X,2HYB,8X,4HDIFF)
410    FORMAT (I11,F10.1,3F10.4)
-REP   2340,2350
       CALL LABELS (LAB,26HEXAMPLE 5
       CALL LABELS (LAB(10),36HINHERENTLY UNSTABLE PROBLEM
-REP   2450,2460
       JOUT=20
       LPRNT=1
-REP   2550
       D1=1.D0
-REP   2590
       DN=0.D0
-REP   2710,2720
       X(1)=1.D0
       X(N)=20.D0
-REP   4380,4410
       ALFA(J)=1.D0/(1.D0+X(J))
       BETA(J)=X(J)**2
       GAMA(J)=0.D0
```

MODIFICATIONS TO RUN EXAMPLE 6

```
-INS   1740
       COMMON /PARM/ P,PS,PI,TP,C1,C2
-INS  1790
       E1=DEXP(-P)
       D=1.D0+E1
       C0=(1.D0-E1)/(P*D)
-REP   1810,1890
       E1=PI*X(K)
       CX=DCOS(E1)
       SX=DSIN(E1)
       CC=CX**2
       CS=CX*SX
       E1=DEXP(P*(X(K)-1.D0))
       E2=DEXP(-P*X(K))
       S1=(E1+E2)/D
       S2=(E1-E2)/D
       A(1)=S1-CC
       A(2)=P*S2+TP*CS
       A(3)=PS*S1+2.D0*C2*CC-C2
       A(4)=S2/P-CS/TP-.5D0*X(K)+C0
       E(1)=DABS(A(1)-Y(K))
       E(2)=DABS(A(2)-F(K))
       E(3)=DABS(A(3)-S(K))
       E(4)=DABS(A(4)-G(K))
-INS   2300
       COMMON /PARM/ P,PS,PI,TP,CA,C2
       DIMENSION PA(4)
       DATA PA/10.D0,15.D0,20.D0,25.D0/
-REP   2320
       IF (JCASE.GT.1) GO TO 10
       PI=4.D0*DATAN(1.D0)
       TP=2.D0*PI
       C2=TP*PI
10     IF (JCASE.GT.4) GO TO 590
        P=PA(JCASE)
       PS=P**2
       CA=PS+2.D0*C2
-REP   2340,2350
       CALL LABELS (LAB,36HEXAMPLE 6
       CALL LABELS (LAB(10),36H
-REP   2450,2460
       JOUT=20
       LPRNT=1
-REP   2590
       DN=0.D0
-INS   2610
       WRITE (6,6004) P
-REP   2720
       X(N)=1.D0
-INS   2890
```

MODIFICATIONS TO RUN EXAMPLE 6

```
6004  FORMAT ('OK =',F10.1)
-INS  4340
      COMMON /PARM/ P,PS,PI,TP,C1,C2
-REP  4380,4410
      ALFA(J)=C1*DCOS(PI*X(J))**2-C2
      BETA(J)=PS
      GAMA(J)=0.D0
```

MODIFICATIONS TO RUN EXAMPLE 7

```
-INS    1740
        COMMON /PARM/ Q,SR
-REP    1810,1850
        AG=SR*(X(K)-.5D0)
        TAG=DTAN(AG)
        SECS=TAG**2+1.D0
        A(2)=2.D0*SR*TAG
        A(3)=Q*SECS
        A(4)=G(K)
        A(1)=DLOG(A(3))
-INS    2300
        COMMON /PARM/ Q,SR
        Q=.5D0
        DO 10 J=1,1000
        P=Q
        SR=DSQRT(.5D0*P)
        Q=DCOS(.5D0*SR)**2
        IF (P.EQ.Q) GO TO 20
10      CONTINUE
        J=1000
20      CONTINUE
-REP    2340,2350
        CALL LABELS (LAB,36HEXAMPLE 7
        CALL LABELS (LAB(10),36HY'' = EXP(Y)
-REP    2450,2460
        JOUT=20
        LPRNT=1
-REP    2590
        DN=0.D0
-INS    2610
        WRITE (6,6000) J,Q,P,Q
-REP    2720
        X(N)=1.D0
-INS    2990
6000    FORMAT (4H0J =,I5,5X,8HLAMBDA =,1PE22.13,10X,2Z20)
-REP    4380,4410
        BETA(J)=DEXP(Y(J))
        ALFA(J)=(1.D0-Y(J))*BETA(J)
        GAMA(J)=0.D0
```

APPENDIX E
PRINTOUTS OF THE EXAMPLE PROBLEMS

```
DC TODD  COPS                         AR008671    10:29   WED MAY 10, 1978
PROBLEM FROM KAMKE

     NDIM      JCASE         N         NT        JOUT      LPRNT      LNORM
      250          1        51         30           0          1          1

  II           TOL         RSC
 0.0          5.00000D-04  1.00000-02

  A           B           C           D
 1.00000 00  0.0         0.0         0.0
 1.00000 00  0.0         0.0         7.00000 00

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE         1        35        51        51  3.560-01  2.31D 15  4.250-01
NEW SPACING COMPUTED
UPDATE         2        34        51        51  1.850-01  1.300 00  2.920-01
NEW SPACING COMPUTED
UPDATE         3        38        51        51  3.410-02  2.780-01  3.870-02
NEW SPACING COMPUTED
UPDATE         4        41        51        51  1.230-03  1.490-02  1.280-03
UPDATE         5        44        51        51  1.250-06  2.710-05  1.790-06

CONVERGED SOLUTION

CHEKDE   51  6.760-10  0.0       0.0       0.0

CHEKBM   11   51   51   15  5.010-04  6.440-04  1.280-03  4.820-04
```

DC TODD COPS AR008671 10:29 WED MAY 10, 1978
PROBLEM FROM KAMKE

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	9.9957D-01	0.0	0.0	1.036D-01	9.89D-01
2	1.0360D-01	1.0393D-01	1.0104D 00	2.0997D-01	5.3740D-03	1.025D-01	9.78D-01
3	2.0608D-01	2.0896D-01	1.0432D 00	4.3589D-01	2.1377D-02	1.003D-01	9.67D-01
4	3.0633D-01	3.1615D-01	1.0995D 00	6.9507D-01	4.7652D-02	9.690D-02	9.54D-01
5	4.0323D-01	4.2641D-01	1.1814D 00	1.0073D 00	8.3563D-02	9.244D-02	9.41D-01
6	4.9566D-01	5.4044D-01	1.2916D 00	1.3958D 00	1.2817D-01	8.698D-02	9.28D-01
7	5.8265D-01	6.5865D-01	1.4334D 00	1.8879D 00	1.8023D-01	8.074D-02	9.16D-01
8	6.6339D-01	7.8117D-01	1.6099D 00	2.5149D 00	2.3826D-01	7.400D-02	9.06D-01
9	7.3738D-01	9.0786D-01	1.8240D 00	3.3115D 00	3.0065D-01	6.702D-02	8.98D-01
10	8.0441D-01	1.0382D 00	2.0779D 00	4.3146D 00	3.6578D-01	6.022D-02	8.93D-01
11	8.6463D-01	1.1719D 00	2.3735D 00	5.5631D 00	4.3223D-01	5.378D-02	8.89D-01
12	9.1841D-01	1.3083D 00	2.7120D 00	7.0969D 00	4.9885D-01	4.784D-02	8.89D-01
13	9.5625D-01	1.4468D 00	3.0939D 00	8.9542D 00	5.6468D-01	4.252D-02	8.89D-01
14	1.0088D 00	1.5871D 00	3.5199D 00	1.1175D 01	6.2912D-01	3.779D-02	8.91D-01
15	1.0466D 00	1.7287D 00	3.9898D 00	1.3798D 01	6.9173D-01	3.366D-02	8.93D-01
16	1.0802D 00	1.8714D 00	4.5038D 00	1.6862D 01	7.5226D-01	3.005D-02	8.96D-01
17	1.1103D 00	2.0148D 00	5.0616D 00	2.0404D 01	8.1060D-01	2.692D-02	8.99D-01
18	1.1372D 00	2.1589D 00	5.6635D 00	2.4463D 01	8.6674D-01	2.419D-02	9.02D-01
19	1.1614D 00	2.3035D 00	6.3090D 00	2.9077D 01	9.2067D-01	2.182D-02	9.06D-01
20	1.1832D 00	2.4485D 00	6.9986D 00	3.4287D 01	9.7249D-01	1.976D-02	9.09D-01
21	1.2030D 00	2.5938D 00	7.7321D 00	4.0131D 01	1.0223D 00	1.796D-02	9.12D-01
22	1.2209D 00	2.7395D 00	8.5096D 00	4.6647D 01	1.0702D 00	1.638D-02	9.15D-01
23	1.2373D 00	2.8855D 00	9.3312D 00	5.3877D 01	1.1162D 00	1.499D-02	9.18D-01
24	1.2523D 00	3.0317D 00	1.0197D 01	6.1860D 01	1.1606D 00	1.376D-02	9.17D-01
25	1.2661D 00	3.1781D 00	1.1107D 01	7.0635D 01	1.2033D 00	1.262D-02	9.22D-01
26	1.2787D 00	3.3241D 00	1.2057D 01	8.0200D 01	1.2443D 00	1.163D-02	9.26D-01
27	1.2903D 00	3.4770D 00	1.3049D 01	9.0609D 01	1.2838D 00	1.078D-02	9.29D-01
28	1.3011D 00	3.6161D 00	1.4085D 01	1.0192D 02	1.3219D 00	1.001D-02	9.31D-01
29	1.3111D 00	3.7623D 00	1.5164D 01	1.1417D 02	1.3588D 00	9.314D-03	9.32D-01
30	1.3204D 00	3.9087D 00	1.6288D 01	1.2740D 02	1.3946D 00	8.685D-03	9.35D-01
31	1.3291D 00	4.0551D 00	1.7455D 01	1.4164D 02	1.4291D 00	8.116D-03	9.36D-01
32	1.3372D 00	4.2016D 00	1.8665D 01	1.5694D 02	1.4626D 00	7.600D-03	9.39D-01
33	1.3448D 00	4.3481D 00	1.9919D 01	1.7332D 02	1.4951D 00	7.132D-03	9.39D-01
34	1.3519D 00	4.4948D 00	2.1216D 01	1.9084D 02	1.5266D 00	6.701D-03	9.41D-01
35	1.3586D 00	4.6413D 00	2.2556D 01	2.0951D 02	1.5572D 00	6.307D-03	9.44D-01
36	1.3649D 00	4.7879D 00	2.3939D 01	2.2937D 02	1.5870D 00	5.952D-03	9.44D-01
37	1.3709D 00	4.9346D 00	2.5366D 01	2.5049D 02	1.6159D 00	5.619D-03	9.46D-01
38	1.3765D 00	5.0812D 00	2.6835D 01	2.7288D 02	1.6440D 00	5.318D-03	9.48D-01
39	1.3818D 00	5.2279D 00	2.8349D 01	2.9659D 02	1.6715D 00	5.041D-03	9.49D-01
40	1.3869D 00	5.3746D 00	2.9906D 01	3.2166D 02	1.6982D 00	4.783D-03	9.51D-01
41	1.3917D 00	5.5215D 00	3.1506D 01	3.4813D 02	1.7242D 00	4.550D-03	9.51D-01
42	1.3962D 00	5.6685D 00	3.3153D 01	3.7609D 02	1.7497D 00	4.329D-03	9.54D-01
43	1.4005D 00	5.8156D 00	3.4844D 01	4.0553D 02	1.7745D 00	4.132D-03	9.54D-01
44	1.4047D 00	5.9632D 00	3.6582D 01	4.3656D 02	1.7989D 00	3.943D-03	9.51D-01
45	1.4086D 00	6.1109D 00	3.8367D 01	4.6920D 02	1.8227D 00	3.751D-03	9.56D-01
46	1.4124D 00	6.2582D 00	4.0190D 01	5.0334D 02	1.8459D 00	3.584D-03	9.58D-01
47	1.4159D 00	6.4055D 00	4.2057D 01	5.3913D 02	1.8686D 00	3.434D-03	9.61D-01
48	1.4194D 00	6.5532D 00	4.3972D 01	5.7667D 02	1.8908D 00	3.300D-03	9.61D-01
49	1.4227D 00	6.7015D 00	4.5939D 01	6.1610D 02	1.9127D 00	3.171D-03	9.64D-01
50	1.4258D 00	6.8503D 00	4.7957D 01	6.5745D 02	1.9342D 00	3.056D-03	
51	1.4289D 00	7.0000D 00	5.0032D 01	7.0089D 02	1.9553D 00		

STOP

DC TODD CDPS　　　　　　　　　　　　ARD00346　　13:38　　MON AUG 07, 1978
EXAMPLE 2　　　　　　　　　　　　　　FALKNER-SKAN EQUATION

K		-0.19884	-0.18000	0.0	0.30000	1.00000	2.00000	10.00000
FO		0.0	0.12864	0.46960	0.77476	1.23259	1.68722	3.67523
	X	Y(X) CORRESPONDING TO ABOVE K VALUES						
1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2	0.1	0.00099	0.01376	0.04696	0.07597	0.11826	0.15876	0.31843
3	0.2	0.00398	0.02933	0.09391	0.14894	0.22661	0.29794	0.54730
4	0.3	0.00895	0.04668	0.14081	0.21886	0.32524	0.41854	0.70496
5	0.4	0.01591	0.06582	0.18761	0.28569	0.41446	0.52190	0.81043
6	0.5	0.02485	0.08673	0.23423	0.34938	0.49465	0.60964	0.87954
7	0.6	0.03578	0.10937	0.28058	0.40988	0.56628	0.68343	0.92414
8	0.7	0.04868	0.13373	0.32653	0.46713	0.62986	0.74496	0.95259
9	0.8	0.06355	0.15975	0.37196	0.52107	0.68594	0.79587	0.97057
10	0.9	0.08038	0.18737	0.41672	0.57167	0.73508	0.83767	0.98185
11	1.0	0.09913	0.21651	0.46063	0.61890	0.77787	0.87172	0.98888
12	1.2	0.14232	0.27899	0.54525	0.70322	0.84567	0.92142	0.99591
13	1.4	0.19274	0.34622	0.62439	0.77425	0.89681	0.95308	0.99856
14	1.6	0.24982	0.41691	0.69670	0.83254	0.93235	0.97269	0.99957
15	1.8	0.31271	0.48946	0.76106	0.87906	0.95683	0.98452	0.99998
16	2.0	0.38026	0.56205	0.81669	0.91509	0.97322	0.99146	0.99999
17	2.2	0.45097	0.63269	0.86330	0.94211	0.98385	0.99542	1.00000
18	2.4	0.52308	0.69942	0.90107	0.96173	0.99055	0.99761	1.00000
19	2.6	0.59460	0.76048	0.93060	0.97548	0.99463	0.99879	1.00000
20	2.8	0.66348	0.81449	0.95288	0.98480	0.99705	0.99940	1.00000
21	3.0	0.72776	0.86061	0.96905	0.99088	0.99842	0.99972	1.00000
22	3.2	0.78578	0.89853	0.98037	0.99471	0.99919	0.99987	1.00000
23	3.4	0.83635	0.92854	0.98797	0.99704	0.99959	0.99995	1.00000
24	3.6	0.87882	0.95138	0.99289	0.99840	0.99980	0.99998	1.00000
25	3.8	0.91315	0.96805	0.99594	0.99916	0.99991	0.99999	1.00000
26	4.0	0.93982	0.97575	0.99777	0.99958	0.99995	1.00000	1.00000
27	4.5	0.97940	0.99449	0.99957	0.99994	0.99999	1.00000	1.00000
28	5.0	0.99439	0.99997	0.99994	0.99999	1.00000	1.00000	1.00000

```
DC TODD  COPS                                     AR00346     13:38    MON AUG 07, 1978
EXAMPLE 2                                         FALKNER-SKAN EQUATION

K =  -0.19884

     NDIM      JCASE          N          NT        JOUT       LPRNT       LNORM
      250          1         51          30          20           0           1

  I1           TOL          RSC
 0.0          5.00000D-04  1.00000-02

  A             B            C            D
 1.0000D 00  0.0          0.0          0.0
 1.0000D 00  0.0          0.0          1.0000D 00

UNIFORM SPACING COMPUTED

UPDATE          1         5          1          41  5.730-01  8.810-01  1.040-01
NEW SPACING COMPUTED
UPDATE          2        11          5          51  2.690-01  4.410-01  7.670-02
NEW SPACING COMPUTED
UPDATE          3        14         20          51  1.390-01  2.230-01  5.140-02
NEW SPACING COMPUTED
UPDATE          4        16         23          51  6.970-02  1.150-01  3.050-02
NEW SPACING COMPUTED
UPDATE          5        16         23          51  3.470-02  5.520-02  1.640-02
NEW SPACING COMPUTED
UPDATE          6        16         23          51  1.740-02  2.730-02  8.380-03
NEW SPACING COMPUTED
UPDATE          7        16         23          51  8.660-03  1.360-02  4.210-03
UPDATE          8        16         23          51  4.280-03  6.750-03  2.090-03
UPDATE          9        16         23          51  2.050-03  3.230-03  1.000-03
UPDATE         10        16         23          51  8.710-04  1.370-03  4.270-04
UPDATE         11        16         23          51  2.450-04  3.860-04  1.200-04

CONVERGED SOLUTION

CHEKDE    16  1.490-06  0.0        0.0        0.0

CHEKBN     2     0     0     0  7.690-02  7.190-04  0.0        0.0

SOLUTION WRITTEN ON UNIT 20
```

```
DC TODD  COPS                                 AR000346    13:38   MON AUG 07, 1978
EXAMPLE 2                                     FALKNER-SKAN EQUATION

K =  -0.18000

     NDIM      JCASE         N         NT        JOUT      LPRNT      LNORM
      250          2        51         30          20          0          1

  I1           TOL         RSC
 0.0           5.00000-04  1.00000-02

  A            B           C            D
 1.00000 00   0.0         0.0          0.0
 1.00000 00   0.0         0.0          1.00000 00

UNIFORM SPACING COMPUTED

UPDATE         1          5          1          46  5.640-01  8.730-01  1.170-01
NEW SPACING COMPUTED
UPDATE         2         12          6          51  2.450-01  4.050-01  8.100-02
NEW SPACING COMPUTED
UPDATE         3         16         23          51  9.820-02  1.610-01  4.190-02
NEW SPACING COMPUTED
UPDATE         4         18         26          51  2.090-02  3.590-02  1.050-02
NEW SPACING COMPUTED
UPDATE         5         19         27          51  1.020-03  1.680-03  5.550-04
UPDATE         6         19         28          51  2.510-06  3.850-06  1.400-06

CONVERGED SOLUTION

CHEKDE   51  1.370-09  0.0       0.0       0.0

CHEKBM   28    0    0    0  1.160-03  5.400-05  0.0       0.0

SOLUTION WRITTEN ON UNIT 20
```

```
DC TODD  CDPS                              AR000346     13:38    MON AUG 07, 1978
EXAMPLE 2                                  FALKNER-SKAN EQUATION

K =    0.0

     NDIM      JCASE          N         NT        JOUT      LPRNT      LNORM
      250          3         51         30          20          0          1

  I1           TOL         RSC
 0.0          5.00000D-04  1.00000D-02

  A           B            C            D
 1.00000D 00  0.0          0.0          0.0
 1.00000D 00  0.0          0.0          1.00000D 00

UNIFORM SPACING COMPUTED

UPDATE         1          5          1         47  4.81D-01  8.04D-01  1.02D-01
NEW SPACING COMPUTED
UPDATE         2         12          5         51  1.24D-01  2.08D-01  4.14D-02
NEW SPACING COMPUTED
UPDATE         3         17         23         51  1.26D-02  1.88D-02  5.11D-03
NEW SPACING COMPUTED
UPDATE         4         20         26         51  1.32D-04  1.98D-04  5.96D-05

CONVERGED SOLUTION

CHEKDE    51  2.21D-06  0.0        0.0        0.0

CHEKBM     3    0    0    0  4.37D-05  3.12D-05  0.0        0.0

SOLUTION WRITTEN ON UNIT 20
```

```
DC TODD  CDPS                                   AR000346     13:38    MON AUG 07, 1978
EXAMPLE 2                                       FALKNER-SKAN EQUATION

K =    0.30000

     NDIM      JCASE          N          NT        JOUT      LPRNT       LNORM
      250          4         51          30          20          0           1

  I1           TOL          RSC
 0.0           5.00000D-04  1.00000D-02

  A            B            C            D
 1.00000D 00   0.0          0.0          0.0
 1.00000D 00   0.0          0.0          1.00000D 00

UNIFORM SPACING COMPUTED

UPDATE          1          5          1          48   3.91D-01   7.17D-01   8.30D-02
NEW SPACING COMPUTED
UPDATE          2         12          3          51   5.99D-02   8.62D-02   1.99D-02
NEW SPACING COMPUTED
UPDATE          3         16          8          51   2.03D-03   1.92D-03   8.05D-04
UPDATE          4         20         27          51   2.43D-06   2.27D-06   1.07D-06

CONVERGED SOLUTION

CHEKDE    50   6.07D-09   0.0       0.0       0.0

CHEKBM     2    0    0    0   6.26D-05   3.88D-06   0.0       0.0

SOLUTION WRITTEN ON UNIT 20
```

```
DC TODD  CD?S                          AR000346     13:38   MON AUG 07, 1978
EXAMPLE 2                              FALKNER-SKAN EQUATION

K =    1.00000

     NDIM      JCASE        N         NT        JOUT      LPRNT      LNORM
      250          5       51         30          20          0          1

  I1            TOL           RSC
 0.0           5.00000D-04  1.00000D-02

  A            B             C             D
 1.00000 00  0.0           0.0           0.0
 1.00000D 00  0.0          0.0           1.00000 00

UNIFORM SPACING COMPUTED

UPDATE          1         5          1          44   2.690-01   5.750-01   5.460-02
NEW SPACING COMPUTED
UPDATE          2        11          1          51   2.160-02   3.280-02   6.550-03
NEW SPACING COMPUTED
UPDATE          3        14          1          51   1.920-04   1.860-04   7.030-05

CONVERGED SOLUTION

CHEKDE    51  3.120-06  0.0        0.0        0.0

CHEKBM     2     0     0     0  4.620-05  1.090-05  0.0        0.0

SOLUTION WRITTEN ON UNIT 20
```

```
DC TODD  CDPS                               ARD00346    13:38   MON AUG 07, 1978
EXAMPLE 2                                   FALKNER-SKAN EQUATION

K =    2.00000

      NDIM       JCASE          N          NT         JOUT      LPRNT      LNORM
       250           6         51          30           20          0          1

  IL            TOL           RSC
 0.0           5.0000D-04   1.0000D-02

  A             B              C              D
 1.00000 00    0.0           0.0            0.0
 1.00000 00    0.0           0.0            1.00000 00

UNIFORM SPACING COMPUTED

UPDATE           1          5            1          39   1.71D-01   4.30D-01   3.46D-02
NEW SPACING COMPUTED
UPDATE           2         10            2          51   7.95D-03   1.25D-02   2.22D-03
UPDATE           3         13            1          51   2.21D-05   2.39D-05   7.33D-06

CONVERGED SOLUTION

CHEKDE    51   3.42D-08   0.0         0.0         0.0

CHEKBM     2     0     0     0   2.42D-05   1.83D-05   0.0         0.0

SOLUTION WRITTEN ON UNIT 20
```

```
DC TODD  COPS                                ARD00346     13:38    MON AUG 07, 1978
EXAMPLE 2                                    FALKNER-SKAN EQUATION

K =  10.00000

     NDIM      JCASE         N          NT        JOUT      LPRNT      LNORM
      250          7        51          30          20          0          1

  I1          TOL          RSC
 0.0         5.00000D-04  1.00000D-02

  A           B            C            D
 1.00000D 00  0.0          0.0          0.0
 1.00000D 00  0.0          0.0          1.00000D 00

UNIFORM SPACING COMPUTED

UPDATE          1          7          1          51  1.180D-01  2.310D-01  3.610D-02
NEW SPACING COMPUTED
UPDATE          2         14          3          51  5.000D-03  5.240D-03  1.390D-03
UPDATE          3         15          1          51  8.890D-06  7.860D-06  2.460D-06

CONVERGED SOLUTION

CHEKDE    51  1.940D-08  0.0        0.0        0.0

CHEKBM    15    0    0    0  1.610D-04  2.530D-05  0.0        0.0

SOLUTION WRITTEN ON UNIT 20

STOP
```

```
DC TODD  COPS                               ARDO0346    13:42   MON AUG 07, 1978
EXAMPLE 3                                   VISCOELASTIC FLUID

K =  1.0000D-03     LAMBDA =  1.0010D 00

    NDIM      JCASE         N         NT        JOUT      LPRNT      LNORM
     250          1        51         30           0          1          1

  I1          TOL          RSC
 0.0          5.0000D-04   1.0000D-02

  A           B            C             D
 1.0000D 00   0.0          0.0           1.0010D 00
 1.0000D 00   0.0          0.0           7.0000D 00

UNIFORM SPACING COMPUTED

UPDATE        1        51        1          1  1.00D 00  2.00D 00  1.00D 00
NEW SPACING COMPUTED
UPDATE        2        51       38         38  5.14D-04  3.03D-03  1.92D-05
UPDATE        3         0        0          0  0.0       0.0       0.0

CONVERGED SOLUTION

CHEKDE    18  1.81D-12  0.0       0.0       7.00D 00

CHEKBM    51    50    51    50  1.29D-03  1.03D-03  2.58D-01  1.80D-05
```

DC TODD CDPS ARD00346 13:42 MON AUG 07, 1978
EXAMPLE 3 VISCOELASTIC FLUID

	X	Y	YP	YPP	I	H	S
1	0.0	1.0010D 00	-1.0020D 00	1.0082D 00	0.0	1.824D-01	1.07D 00
2	1.8242D-01	8.3398D-01	-8.3481D-01	8.3044D-01	1.6690D-01	1.945D-01	1.06D 00
3	3.7691D-01	6.8639D-01	-6.8709D-01	6.9366D-01	3.1428D-01	2.068D-01	1.05D 00
4	5.8371D-01	5.5810D-01	-5.5865D-01	5.5314D-01	4.4251D-01	2.177D-01	1.04D 00
5	8.0138D-01	4.4876D-01	-4.4922D-01	4.5643D-01	5.5166D-01	2.262D-01	1.03D 00
6	1.0276D 00	3.5791D-01	-3.5826D-01	3.5145D-01	6.4251D-01	2.324D-01	1.02D 00
7	1.2600D 00	2.8352D-01	-2.8381D-01	2.9192D-01	7.1672D-01	2.367D-01	1.01D 00
8	1.4967D 00	2.2382D-01	-2.2403D-01	2.1589D-01	7.7648D-01	2.395D-01	1.01D 00
9	1.7362D 00	1.7600D-01	-1.7618D-01	1.8539D-01	8.2413D-01	2.413D-01	1.00D 00
10	1.9775D 00	1.3835D-01	-1.3848D-01	1.2894D-01	8.6187D-01	2.424D-01	1.00D 00
11	2.2199D 00	1.0841D-01	-1.0853D-01	1.1905D-01	8.9163D-01	2.431D-01	1.00D 00
12	2.4630D 00	8.5134D-02	-8.5208D-02	7.4127D-02	9.1504D-01	2.436D-01	1.00D 00
13	2.7066D 00	6.6560D-02	-6.6639D-02	7.8699D-02	9.3343D-01	2.438D-01	1.00D 00
14	2.9504D 00	5.2309D-02	-5.2348D-02	3.9538D-02	9.4785D-01	2.440D-01	1.00D 00
15	3.1944D 00	4.0796D-02	-4.0851D-02	5.4698D-02	9.5915D-01	2.441D-01	1.00D 00
16	3.4385D 00	3.2141D-02	-3.2158D-02	1.7383D-02	9.6801D-01	2.442D-01	1.00D 00
17	3.6827D 00	2.4968D-02	-2.5009D-02	4.0924D-02	9.7495D-01	2.442D-01	1.00D 00
18	3.9269D 00	1.9771D-02	-1.9774D-02	2.7498D-03	9.8038D-01	2.442D-01	1.00D 00
19	4.1712D 00	1.5249D-02	-1.5283D-02	3.3585D-02	9.8464D-01	2.443D-01	1.00D 00
20	4.4154D 00	1.2192D-02	-1.2185D-02	-7.4190D-03	9.8797D-01	2.443D-01	1.00D 00
21	4.6597D 00	9.2786D-03	-9.3089D-03	3.0363D-02	9.9058D-01	2.443D-01	1.00D 00
22	4.9040D 00	7.5544D-03	-7.5394D-03	-1.5029D-02	9.9263D-01	2.443D-01	1.00D 00
23	5.1482D 00	5.6061D-03	-5.6360D-03	2.9861D-02	9.9423D-01	2.443D-01	1.00D 00
24	5.3925D 00	4.7221D-03	-4.7008D-03	-2.1275D-02	9.9548D-01	2.443D-01	1.00D 00
25	5.6368D 00	3.3415D-03	-3.3728D-03	3.1248D-02	9.9646D-01	2.443D-01	1.00D 00
26	5.8811D 00	2.9988D-03	-2.9719D-03	-2.6925D-02	9.9723D-01	2.443D-01	1.00D 00
27	6.1253D 00	1.9383D-03	-1.9724D-03	3.4050D-02	9.9783D-01	2.443D-01	1.00D 00
28	6.3696D 00	1.9577D-03	-1.9252D-03	-3.2483D-02	9.9831D-01	2.443D-01	1.00D 00
29	6.6139D 00	1.0612D-03	-1.0992D-03	3.8015D-02	9.9867D-01	2.443D-01	1.00D 00
30	6.8582D 00	1.3371D-03	-1.2988D-03	-3.8300D-02	9.9897D-01	2.443D-01	1.00D 00
31	7.1025D 00	5.0410D-04	-5.4713D-04	4.3030D-02	9.9919D-01	2.443D-01	1.00D 00
32	7.3467D 00	9.7725D-04	-9.3261D-04	-4.4640D-02	9.9937D-01	2.443D-01	1.00D 00
33	7.5910D 00	1.4037D-04	-1.8945D-04	4.9080D-02	9.9950D-01	2.443D-01	1.00D 00
34	7.8353D 00	7.8028D-04	-7.2856D-04	-5.1718D-02	9.9962D-01	2.443D-01	1.00D 00
35	8.0796D 00	-1.0812D-04	5.1912D-05	5.6212D-02	9.9970D-01	2.443D-01	1.00D 00
36	8.3238D 00	6.8681D-04	-6.2707D-04	-5.9731D-02	9.9977D-01	2.443D-01	1.00D 00
37	8.5681D 00	-2.8979D-04	2.2527D-04	6.4525D-02	9.9981D-01	2.443D-01	1.00D 00
38	6.8124D 00	6.6091D-04	-5.9204D-04	-6.8870D-02	9.9986D-01	2.443D-01	1.00D 00
39	9.0567D 00	-4.3490D-04	3.6075D-04	7.4156D-02	9.9989D-01	2.443D-01	1.00D 00
40	9.3010D 00	6.8119D-04	-6.0185D-04	-7.9338D-02	9.9992D-01	2.443D-01	1.00D 00
41	9.5452D 00	-5.6268D-04	4.7740D-04	8.5279D-02	9.9993D-01	2.443D-01	1.00D 00
42	9.7895D 00	7.3525D-04	-6.4389D-04	-9.1353D-02	9.9996D-01	2.443D-01	1.00D 00
43	1.0034D 01	-6.8568D-04	5.8757D-04	9.8104D-02	9.9996D-01	2.443D-01	1.00D 00
44	1.0278D 01	8.1628D-04	-7.1112D-04	-1.0516D-01	9.9998D-01	2.443D-01	1.00D 00
45	1.0522D 01	-8.1248D-04	6.9960D-04	1.1288D-01	9.9997D-01	2.443D-01	1.00D 00
46	1.0767D 01	9.2109D-04	-8.0005D-04	-1.2104D-01	9.9999D-01	2.443D-01	1.00D 00
47	1.1011D 01	-9.4936D-04	8.1947D-04	1.2989D-01	9.9998D-01	2.443D-01	1.00D 00
48	1.1255D 01	1.0488D-03	-9.0948D-04	-1.3931D-01	1.0000D 00	2.443D-01	1.00D 00
49	1.1499D 01	-1.1013D-03	9.5186D-04	1.4947D-01	9.9999D-01	2.443D-01	1.00D 00
50	1.1744D 01	1.2001D-03	-1.0397D-03	-1.6038D-01	1.0000D 00	2.443D-01	
51	1.1988D 01	-1.2793D-03	1.0217D-03	2.5753D-01	1.0000D 00		

```
DC TODD  CDPS                                    AR000346     13:42   MON AUG 07, 1978
EXAMPLE 3                                        VISCOELASTIC FLUID

K =  1.0000D-02      LAMBDA =  1.0102D 00

     NDIM      JCASE          N          NT        JOUT      LPRNT      LNORM
      250          2         51          30           0          1          1

  II          TOL          RSC
 0.0          5.0000D-04   1.0000D-02

  A            B            C            D
 1.0000D 00  0.0          0.0          1.0102D 00
 1.0000D 00  0.0          0.0          7.0000D 00

UNIFORM SPACING COMPUTED

UPDATE          1        51         1          1  1.00D 00  2.02D 00  1.00D 00
NEW SPACING COMPUTED
UPDATE          2         8         3          3  4.27D-05  2.82D-04  2.09D-06

CONVERGED SOLUTION

CHEKDE    26   8.06D-13  0.0       0.0       7.00D 00

CHEKBM     2     1     1     7  3.75D-05  5.63D-05  5.63D-03  1.07D-05
```

DC TODD COPS　　　　AR000346　13:42　MON AUG 07, 1978
EXAMPLE 3　　　　VISCOELASTIC FLUID

	X	Y	YP	YPP	I	H	S
1	0.0	1.0102D 00	-1.0206D 00	1.0366D 00	0.0	1.794D-01	1.07D 00
2	1.7939D-01	8.4280D-01	-8.5135D-01	8.5531D-01	1.6575D-01	1.914D-01	1.06D 00
3	3.7077D-01	6.9460D-01	-7.0174D-01	7.1370D-01	3.1241D-01	2.038D-01	1.05D 00
4	5.7452D-01	5.6543D-01	-5.7116D-01	5.7264D-01	4.4032D-01	2.148D-01	1.04D 00
5	7.8930D-01	4.5509D-01	-4.5978D-01	4.6889D-01	5.4949D-01	2.235D-01	1.03D 00
6	1.0128D 00	3.6315D-01	-3.6682D-01	3.6641D-01	6.4056D-01	2.299D-01	1.02D 00
7	1.2428D 00	2.8783D-01	-2.9081D-01	2.9790D-01	7.1506D-01	2.343D-01	1.01D 00
8	1.4771D 00	2.2721D-01	-2.2949D-01	2.2787D-01	7.7512D-01	2.372D-01	1.01D 00
9	1.7143D 00	1.7874D-01	-1.8060D-01	1.8633D-01	8.2305D-01	2.391D-01	1.00D 00
10	1.9534D 00	1.4043D-01	-1.4183D-01	1.3953D-01	8.6102D-01	2.403D-01	1.00D 00
11	2.1937D 00	1.1012D-01	-1.1128D-01	1.1608D-01	8.9097D-01	2.410D-01	1.00D 00
12	2.4347D 00	8.6370D-02	-8.7216D-02	8.4561D-02	9.1454D-01	2.415D-01	1.00D 00
13	2.6762D 00	6.7628D-02	-6.8353D-02	7.2503D-02	9.3304D-01	2.418D-01	1.00D 00
14	2.9179D 00	5.3019D-02	-5.3526D-02	5.0727D-02	9.4755D-01	2.419D-01	1.00D 00
15	3.1598D 00	4.1480D-02	-4.1936D-02	4.5618D-02	9.5892D-01	2.420D-01	1.00D 00
16	3.4019D 00	3.2524D-02	-3.2824D-02	3.0003D-02	9.6783D-01	2.421D-01	1.00D 00
17	3.6440D 00	2.5427D-02	-2.5717D-02	2.9047D-02	9.7481D-01	2.421D-01	1.00D 00
18	3.8861D 00	1.9949D-02	-2.0122D-02	1.7351D-02	9.8028D-01	2.422D-01	1.00D 00
19	4.1283D 00	1.5581D-02	-1.5770D-02	1.8822D-02	9.8456D-01	2.422D-01	1.00D 00
20	4.3705D 00	1.2237D-02	-1.2334D-02	9.6526D-03	9.8791D-01	2.422D-01	1.00D 00
21	4.6127D 00	9.5450D-03	-9.6699D-03	1.2495D-02	9.9054D-01	2.422D-01	1.00D 00
22	4.8549D 00	7.5085D-03	-7.5584D-03	4.9888D-03	9.9259D-01	2.422D-01	1.00D 00
23	5.0971D 00	5.8447D-03	-5.9303D-03	8.5615D-03	9.9420D-01	2.422D-01	1.00D 00
24	5.3393D 00	4.6093D-03	-4.6311D-03	2.1827D-03	9.9546D-01	2.422D-01	1.00D 00
25	5.5815D 00	3.5767D-03	-3.6377D-03	6.0986D-03	9.9645D-01	2.422D-01	1.00D 00
26	5.8237D 00	2.8316D-03	-2.8368D-03	5.1251D-04	9.9722D-01	2.422D-01	1.00D 00
27	6.0659D 00	2.1867D-03	-2.2321D-03	4.5401D-03	9.9782D-01	2.422D-01	1.00D 00
28	6.3081D 00	1.7415D-03	-1.7369D-03	-4.6421D-04	9.9830D-01	2.422D-01	1.00D 00
29	6.5503D 00	1.3350D-03	-1.3703D-03	3.5390D-03	9.9867D-01	2.422D-01	1.00D 00
30	6.7925D 00	1.0730D-03	-1.0628D-03	-1.0185D-03	9.9896D-01	2.422D-01	1.00D 00
31	7.0347D 00	8.1312D-04	-8.4195D-04	2.8821D-03	9.9919D-01	2.422D-01	1.00D 00
32	7.2769D 00	6.6282D-04	-6.4966D-04	-1.3164D-03	9.9936D-01	2.422D-01	1.00D 00
33	7.5191D 00	4.9353D-04	-5.1792D-04	2.4387D-03	9.9950D-01	2.422D-01	1.00D 00
34	7.7613D 00	4.1110D-04	-3.9651D-04	-1.4594D-03	9.9961D-01	2.422D-01	1.00D 00
35	8.0035D 00	2.9790D-04	-3.1919D-04	2.1286D-03	9.9970D-01	2.422D-01	1.00D 00
36	8.2457D 00	2.5652D-04	-2.4142D-04	-1.5099D-03	9.9976D-01	2.422D-01	1.00D 00
37	8.4879D 00	1.7824D-04	-1.9726D-04	1.9022D-03	9.9982D-01	2.422D-01	1.00D 00
38	8.7301D 00	1.6150D-04	-1.4645D-04	-1.5056D-03	9.9986D-01	2.422D-01	1.00D 00
39	8.9723D 00	1.0514D-04	-1.2243D-04	1.7294D-03	9.9989D-01	2.422D-01	1.00D 00
40	9.2145D 00	1.0301D-04	-8.8314D-05	-1.4699D-03	9.9992D-01	2.422D-01	1.00D 00
41	9.4567D 00	6.0558D-05	-7.6470D-05	1.5912D-03	9.9994D-01	2.422D-01	1.00D 00
42	9.6989D 00	6.6928D-05	-5.2761D-05	-1.4168D-03	9.9995D-01	2.422D-01	1.00D 00
43	9.9411D 00	3.3453D-05	-4.8215D-05	1.4762D-03	9.9996D-01	2.422D-01	1.00D 00
44	1.0183D 01	4.4592D-05	-3.1044D-05	-1.3548D-03	9.9997D-01	2.422D-01	1.00D 00
45	1.0426D 01	1.7047D-05	-3.0818D-05	1.3771D-03	9.9998D-01	2.422D-01	1.00D 00
46	1.0668D 01	3.0696D-05	-1.7806D-05	-1.2890D-03	9.9999D-01	2.422D-01	1.00D 00
47	1.0910D 01	7.1874D-06	-2.0081D-05	1.2894D-03	9.9999D-01	2.422D-01	1.00D 00
48	1.1152D 01	2.1985D-05	-9.7608D-06	-1.2224D-03	9.9999D-01	2.422D-01	1.00D 00
49	1.1394D 01	1.3296D-06	-1.3431D-05	1.2102D-03	1.0000D 00	2.422D-01	1.00D 00
50	1.1637D 01	1.6467D-05	-4.8653D-06	-1.1601D-03	1.0000D 00	2.422D-01	
51	1.1879D 01	-2.4690D-06	-1.4158D-05	1.6627D-03	1.0000D 00		

```
DC TODD  COPS                              AR000346    13:42   MON AUG 07, 1978
EXAMPLE 3                                  VISCOELASTIC FLUID

K =  1.0000D-01     LAMBDA =  1.1270D 00

     NDIM      JCASE          N          NT       JOUT      LPRNT      LNORM
      250          3         51          30          0          1          1

  I1           TOL          RSC
 0.0          5.0000D-04  1.0000D-02

  A            B            C            D
 1.0000D 00  0.0          0.0          1.1270D 00
 1.0000D 00  0.0          0.0          7.0000D 00

UNIFORM SPACING COMPUTED

UPDATE          1          51         1          1  1.000D 00  2.270D 00  1.000D 00
NEW SPACING COMPUTED
UPDATE          2           2         1          5  2.020D-05  5.800D-04  4.970D-06
UPDATE          3           0         0          0  0.0        0.0        0.0

CONVERGED SOLUTION

CHEKDE    26  1.250-14  0.0        0.0        7.000 00

CHEKBN     3    1    1    7  2.660-05  1.670-04  1.670-03  1.550-05
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DC TODD CDPS ARD00346 13:42 MON AUG 07, 1978
EXAMPLE 3 VISCOELASTIC FLUID

	X	Y	YP	YPP	I	H	S
1	0.0	1.1270D 00	-1.2700D 00	1.4298D 00	0.0	1.453D-01	1.07D 00
2	1.4527D-01	9.5684D-01	-1.0782D 00	1.2139D 00	1.5102D-01	1.552D-01	1.08D 00
3	3.0044D-01	8.0332D-01	-9.0537D-01	1.0205D 00	2.8724D-01	1.675D-01	1.08D 00
4	4.6795D-01	6.6512D-01	-7.4962D-01	8.4494D-01	4.0986D-01	1.802D-01	1.07D 00
5	6.4815D-01	5.4287D-01	-6.1185D-01	6.8975D-01	5.1833D-01	1.920D-01	1.04D 00
6	8.4016D-01	4.3723D-01	-4.9278D-01	5.5550D-01	6.1206D-01	2.006D-01	1.03D 00
7	1.0407D 00	3.4877D-01	-3.9308D-01	4.4310D-01	6.9055D-01	2.067D-01	1.02D 00
8	1.2474D 00	2.7630D-01	-3.1140D-01	3.5101D-01	7.5486D-01	2.110D-01	1.01D 00
9	1.4585D 00	2.1781D-01	-2.4548D-01	2.7669D-01	8.0675D-01	2.140D-01	1.01D 00
10	1.6724D 00	1.7113D-01	-1.9287D-01	2.1739D-01	8.4817D-01	2.159D-01	1.01D 00
11	1.8884D 00	1.3417D-01	-1.5121D-01	1.7043D-01	8.8096D-01	2.171D-01	1.00D 00
12	2.1055D 00	1.0504D-01	-1.1839D-01	1.3343D-01	9.0680D-01	2.179D-01	1.00D 00
13	2.3234D 00	8.2171D-02	-9.2609D-02	1.0438D-01	9.2710D-01	2.184D-01	1.00D 00
14	2.5417D 00	6.4244D-02	-7.2405D-02	8.1607D-02	9.4300D-01	2.187D-01	1.00D 00
15	2.7604D 00	5.0212D-02	-5.6591D-02	6.3783D-02	9.5545D-01	2.188D-01	1.00D 00
16	2.9792D 00	3.9237D-02	-4.4222D-02	4.9842D-02	9.6519D-01	2.189D-01	1.00D 00
17	3.1982D 00	3.0657D-02	-3.4552D-02	3.8943D-02	9.7280D-01	2.190D-01	1.00D 00
18	3.4172D 00	2.3952D-02	-2.6994D-02	3.0425D-02	9.7875D-01	2.190D-01	1.00D 00
19	3.6362D 00	1.8712D-02	-2.1089D-02	2.3769D-02	9.8340D-01	2.191D-01	1.00D 00
20	3.8553D 00	1.4618D-02	-1.6475D-02	1.8569D-02	9.8704D-01	2.191D-01	1.00D 00
21	4.0744D 00	1.1420D-02	-1.2870D-02	1.4506D-02	9.8987D-01	2.191D-01	1.00D 00
22	4.2935D 00	8.9211D-03	-1.0054D-02	1.1332D-02	9.9209D-01	2.191D-01	1.00D 00
23	4.5126D 00	6.9691D-03	-7.8543D-03	8.8525D-03	9.9382D-01	2.191D-01	1.00D 00
24	4.7317D 00	5.4442D-03	-6.1357D-03	6.9154D-03	9.9518D-01	2.191D-01	1.00D 00
25	4.9508D 00	4.2529D-03	-4.7931D-03	5.4022D-03	9.9623D-01	2.191D-01	1.00D 00
26	5.1699D 00	3.3223D-03	-3.7443D-03	4.2202D-03	9.9706D-01	2.191D-01	1.00D 00
27	5.3890D 00	2.5953D-03	-2.9250D-03	3.2967D-03	9.9770D-01	2.191D-01	1.00D 00
28	5.6081D 00	2.0274D-03	-2.2850D-03	2.5753D-03	9.9821D-01	2.191D-01	1.00D 00
29	5.8272D 00	1.5838D-03	-1.7850D-03	2.0118D-03	9.9860D-01	2.191D-01	1.00D 00
30	6.0463D 00	1.2372D-03	-1.3944D-03	1.5716D-03	9.9891D-01	2.191D-01	1.00D 00
31	6.2654D 00	9.6651D-04	-1.0893D-03	1.2277D-03	9.9915D-01	2.191D-01	1.00D 00
32	6.4845D 00	7.5502D-04	-8.5092D-04	9.5906D-04	9.9934D-01	2.191D-01	1.00D 00
33	6.7036D 00	5.8981D-04	-6.6473D-04	7.4920D-04	9.9948D-01	2.191D-01	1.00D 00
34	6.9227D 00	4.6075D-04	-5.1927D-04	5.8527D-04	9.9960D-01	2.191D-01	1.00D 00
35	7.1418D 00	3.5993D-04	-4.0565D-04	4.5720D-04	9.9969D-01	2.191D-01	1.00D 00
36	7.3610D 00	2.8117D-04	-3.1689D-04	3.5716D-04	9.9976D-01	2.191D-01	1.00D 00
37	7.5801D 00	2.1965D-04	-2.4755D-04	2.7900D-04	9.9981D-01	2.191D-01	1.00D 00
38	7.7992D 00	1.7158D-04	-1.9338D-04	2.1795D-04	9.9985D-01	2.191D-01	1.00D 00
39	8.0183D 00	1.3404D-04	-1.5106D-04	1.7026D-04	9.9989D-01	2.191D-01	1.00D 00
40	8.2374D 00	1.0471D-04	-1.1801D-04	1.3301D-04	9.9991D-01	2.191D-01	1.00D 00
41	8.4565D 00	8.1796D-05	-9.2186D-05	1.0390D-04	9.9993D-01	2.191D-01	1.00D 00
42	8.6756D 00	6.3898D-05	-7.2015D-05	8.1166D-05	9.9995D-01	2.191D-01	1.00D 00
43	8.8947D 00	4.9916D-05	-5.6257D-05	6.3406D-05	9.9996D-01	2.191D-01	1.00D 00
44	9.1138D 00	3.8994D-05	-4.3947D-05	4.9532D-05	9.9997D-01	2.191D-01	1.00D 00
45	9.3329D 00	3.0461D-05	-3.4330D-05	3.8693D-05	9.9998D-01	2.191D-01	1.00D 00
46	9.5520D 00	2.3796D-05	-2.6818D-05	3.0226D-05	9.9999D-01	2.191D-01	1.00D 00
47	9.7711D 00	1.8589D-05	-2.0950D-05	2.3612D-05	9.9999D-01	2.191D-01	1.00D 00
48	9.9903D 00	1.4521D-05	-1.6366D-05	1.8446D-05	9.9999D-01	2.191D-01	1.00D 00
49	1.0209D 01	1.1344D-05	-1.2785D-05	1.4409D-05	1.0000D 00	2.191D-01	1.00D 00
50	1.0428D 01	8.8615D-06	-9.9873D-06	1.1257D-05	1.0000D 00	2.191D-01	
51	1.0648D 01	6.9226D-06	-7.7996D-06	8.7703D-06	1.0000D 00		

STOP

```
DC TODD  COPS                              AR000346     13:45    MON AUG 07, 1978
EXAMPLE 4                                  CHEMICAL DISPERSION

K =    1.00000

     NDIM      JCASE         N         NT        JOUT      LPRNT      LNORM
      250         1         51         30          20          1          1

  IL          TOL          RSC
 0.0         5.00000D-04  1.00000-02

  A            B            C           D
 1.00000 00 -1.00000D 00  0.0          1.0000D 00
 0.0          1.00000 00  0.0          0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE          1          51        1         51   1.23D 00   4.23D-01   3.33D 00
NEW SPACING COMPUTED
UPDATE          2          51        7         51   4.04D-02   1.28D-02   1.45D-02
NEW SPACING COMPUTED
UPDATE          3          51       13         51   1.39D-05   4.39D-06   4.73D-06

CONVERGED SOLUTION

CHECKDE    39  1.95D-12  0.0        5.72D-15  2.26D-15

CHEKBM     0     0     0     0  5.07D-08  1.28D-06  7.25D-07  0.0

SOLUTION WRITTEN ON UNIT 20
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DC TODD CDPS   AROO 0346   13:45   MON AUG 07, 1978
EXAMPLE 4   CHEMICAL DISPERSION

	X	Y	YP	YPP	I	H	S
1	0.0	6.2004D-01	-3.7996D-01	3.4984D-01	0.0	1.919D-02	1.000 00
2	1.9194D-02	6.1281D-01	-3.7326D-01	3.4853D-01	1.1832D-02	1.924D-02	1.000 00
3	3.8431D-02	6.0570D-01	-3.6656D-01	3.4732D-01	2.3552D-02	1.928D-02	1.000 00
4	5.7709D-02	5.9870D-01	-3.5988D-01	3.4622D-01	3.5161D-02	1.932D-02	1.000 00
5	7.7029D-02	5.9181D-01	-3.5320D-01	3.4523D-01	4.6660D-02	1.936D-02	1.000 00
6	9.6389D-02	5.8503D-01	-3.4652D-01	3.4435D-01	5.8052D-02	1.940D-02	1.000 00
7	1.1579D-01	5.7838D-01	-3.3985D-01	3.4359D-01	6.9337D-02	1.944D-02	1.000 00
8	1.3523D-01	5.7183D-01	-3.3318D-01	3.4295D-01	8.0516D-02	1.948D-02	1.000 00
9	1.5471D-01	5.6541D-01	-3.2650D-01	3.4244D-01	9.1592D-02	1.952D-02	1.000 00
10	1.7422D-01	5.5910D-01	-3.1982D-01	3.4205D-01	1.0257D-01	1.955D-02	1.000 00
11	1.9378D-01	5.5291D-01	-3.1314D-01	3.4179D-01	1.1344D-01	1.959D-02	1.000 00
12	2.1337D-01	5.4684D-01	-3.0644D-01	3.4167D-01	1.2421D-01	1.963D-02	1.000 00
13	2.3300D-01	5.4090D-01	-2.9974D-01	3.4169D-01	1.3489D-01	1.967D-02	1.000 00
14	2.5266D-01	5.3507D-01	-2.9302D-01	3.4185D-01	1.4546D-01	1.970D-02	1.000 00
15	2.7236D-01	5.2936D-01	-2.8628D-01	3.4216D-01	1.5595D-01	1.974D-02	1.000 00
16	2.9210D-01	5.2378D-01	-2.7952D-01	3.4261D-01	1.6634D-01	1.977D-02	1.000 00
17	3.1187D-01	5.1832D-01	-2.7274D-01	3.4323D-01	1.7664D-01	1.980D-02	1.000 00
18	3.3168D-01	5.1298D-01	-2.6594D-01	3.4400D-01	1.8686D-01	1.984D-02	1.000 00
19	3.5151D-01	5.0778D-01	-2.5910D-01	3.4494D-01	1.9698D-01	1.987D-02	1.000 00
20	3.7139D-01	5.0270D-01	-2.5224D-01	3.4605D-01	2.0702D-01	1.990D-02	1.000 00
21	3.9129D-01	4.9774D-01	-2.4534D-01	3.4734D-01	2.1698D-01	1.994D-02	1.000 00
22	4.1122D-01	4.9292D-01	-2.3840D-01	3.4881D-01	2.2685D-01	1.997D-02	1.000 00
23	4.3119D-01	4.8823D-01	-2.3142D-01	3.5046D-01	2.3665D-01	2.000D-02	1.000 00
24	4.5119D-01	4.8367D-01	-2.2439D-01	3.5231D-01	2.4636D-01	2.003D-02	1.000 00
25	4.7122D-01	4.7925D-01	-2.1732D-01	3.5435D-01	2.5601D-01	2.006D-02	1.000 00
26	4.9127D-01	4.7496D-01	-2.1019D-01	3.5660D-01	2.6557D-01	2.009D-02	1.000 00
27	5.1136D-01	4.7081D-01	-2.0300D-01	3.5905D-01	2.7507D-01	2.011D-02	1.000 00
28	5.3148D-01	4.6680D-01	-1.9575D-01	3.6173D-01	2.8450D-01	2.014D-02	1.000 00
29	5.5162D-01	4.6293D-01	-1.8844D-01	3.6463D-01	2.9387D-01	2.017D-02	1.000 00
30	5.7179D-01	4.5921D-01	-1.8105D-01	3.6776D-01	3.0316D-01	2.020D-02	1.000 00
31	5.9198D-01	4.5563D-01	-1.7359D-01	3.7112D-01	3.1240D-01	2.022D-02	1.000 00
32	6.1220D-01	4.5219D-01	-1.6605D-01	3.7474D-01	3.2158D-01	2.025D-02	1.000 00
33	6.3245D-01	4.4891D-01	-1.5842D-01	3.7860D-01	3.3070D-01	2.027D-02	1.000 00
34	6.5272D-01	4.4577D-01	-1.5071D-01	3.8273D-01	3.3977D-01	2.029D-02	1.000 00
35	6.7301D-01	4.4279D-01	-1.4290D-01	3.8713D-01	3.4879D-01	2.032D-02	1.000 00
36	6.9333D-01	4.3997D-01	-1.3499D-01	3.9180D-01	3.5775D-01	2.034D-02	1.000 00
37	7.1366D-01	4.3731D-01	-1.2697D-01	3.9676D-01	3.6667D-01	2.036D-02	1.000 00
38	7.3402D-01	4.3481D-01	-1.1884D-01	4.0202D-01	3.7555D-01	2.038D-02	1.000 00
39	7.5440D-01	4.3247D-01	-1.1059D-01	4.0759D-01	3.8438D-01	2.040D-02	1.000 00
40	7.7479D-01	4.3030D-01	-1.0222D-01	4.1347D-01	3.9318D-01	2.041D-02	1.000 00
41	7.9521D-01	4.2830D-01	-9.3714D-02	4.1967D-01	4.0195D-01	2.043D-02	1.000 00
42	8.1564D-01	4.2647D-01	-8.5075D-02	4.2621D-01	4.1068D-01	2.044D-02	1.000 00
43	8.3608D-01	4.2482D-01	-7.6292D-02	4.3310D-01	4.1938D-01	2.046D-02	1.000 00
44	8.5654D-01	4.2335D-01	-6.7358D-02	4.4034D-01	4.2805D-01	2.047D-02	1.000 00
45	8.7701D-01	4.2207D-01	-5.8267D-02	4.4795D-01	4.3671D-01	2.048D-02	1.000 00
46	8.9749D-01	4.2097D-01	-4.9011D-02	4.5594D-01	4.4534D-01	2.049D-02	1.000 00
47	9.1798D-01	4.2006D-01	-3.9583D-02	4.6432D-01	4.5395D-01	2.050D-02	1.000 00
48	9.3848D-01	4.1935D-01	-2.9976D-02	4.7311D-01	4.6256D-01	2.050D-02	1.000 00
49	9.5898D-01	4.1883D-01	-2.0182D-02	4.8231D-01	4.7115D-01	2.051D-02	1.000 00
50	9.7949D-01	4.1852D-01	-1.0193D-02	4.9194D-01	4.7974D-01	2.051D-02	
51	1.0000D 00	4.1841D-01	2.2612D-15	5.0201D-01	4.8832D-01		

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DC TODO  COPS                                  AR000346     13:45    MON AUG 07, 1978
EXAMPLE 4                                      CHEMICAL DISPERSION

K =    0.10000

     NDIM       JCASE          N          NT        JOUT       LPRNT       LNORM
      250          2          51          30          20           1           1

  I1           TOL          RSC
 0.0          5.00000-04  1.00000-02

  A            B            C           D
 1.00000 00 -1.00000-01  0.0          1.00000 00
 0.0          1.00000 00  0.0          0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE          1          15          1          51  1.340-01  3.740-01  8.370-02
NEW SPACING COMPUTED
UPDATE          2          50         11          51  2.710-04  3.750-04  2.090-04

CONVERGED SOLUTION

CHECKDE    32  3.730-08  0.0        3.610-16  1.830-16

CHEKBN     0     0     0     0  4.790-08  6.340-07  4.480-05  0.0

SOLUTION WRITTEN ON UNIT 20
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DC 1000 CDPS AR000346 13:45 MON AUG 07, 1978
EXAMPLE 4 CHEMICAL DISPERSION

	X	Y	YP	YPP	I	H	S
1	0.0	9.0541D-01	-9.4593D-01	9.1866D-01	0.0	1.701D-02	1.01D 00
2	1.7006D-02	8.8945D-01	-9.3042D-01	9.0587D-01	1.5262D-02	1.713D-02	1.01D 00
3	3.4139D-02	8.7364D-01	-9.1500D-01	8.9309D-01	3.0365D-02	1.726D-02	1.01D 00
4	5.1400D-02	8.5798D-01	-8.9970D-01	8.8033D-01	4.5309D-02	1.739D-02	1.01D 00
5	6.8788D-02	8.4247D-01	-8.8450D-01	8.6760D-01	6.0092D-02	1.752D-02	1.01D 00
6	8.6304D-02	8.2711D-01	-8.6942D-01	8.5489D-01	7.4714D-02	1.765D-02	1.01D 00
7	1.0395D-01	8.1190D-01	-8.5444D-01	8.4222D-01	8.9174D-02	1.777D-02	1.01D 00
8	1.2172D-01	7.9685D-01	-8.3959D-01	8.2958D-01	1.0347D-01	1.791D-02	1.01D 00
9	1.3963D-01	7.8195D-01	-8.2485D-01	8.1699D-01	1.1761D-01	1.805D-02	1.01D 00
10	1.5768D-01	7.6719D-01	-8.1022D-01	8.0444D-01	1.3158D-01	1.817D-02	1.01D 00
11	1.7585D-01	7.5260D-01	-7.9571D-01	7.9195D-01	1.4539D-01	1.830D-02	1.01D 00
12	1.9415D-01	7.3817D-01	-7.8133D-01	7.7952D-01	1.5903D-01	1.843D-02	1.01D 00
13	2.1258D-01	7.2390D-01	-7.6708D-01	7.6717D-01	1.7250D-01	1.855D-02	1.01D 00
14	2.3113D-01	7.0980D-01	-7.5296D-01	7.5490D-01	1.8580D-01	1.868D-02	1.01D 00
15	2.4981D-01	6.9587D-01	-7.3898D-01	7.4272D-01	1.9893D-01	1.880D-02	1.01D 00
16	2.6861D-01	6.8211D-01	-7.2513D-01	7.3065D-01	2.1188D-01	1.892D-02	1.01D 00
17	2.8753D-01	6.6852D-01	-7.1141D-01	7.1869D-01	2.2466D-01	1.905D-02	1.01D 00
18	3.0658D-01	6.5510D-01	-6.9784D-01	7.0688D-01	2.3727D-01	1.917D-02	1.01D 00
19	3.2575D-01	6.4185D-01	-6.8440D-01	6.9521D-01	2.4970D-01	1.929D-02	1.01D 00
20	3.4504D-01	6.2878D-01	-6.7110D-01	6.8373D-01	2.6195D-01	1.941D-02	1.01D 00
21	3.6444D-01	6.1588D-01	-6.5794D-01	6.7246D-01	2.7403D-01	1.952D-02	1.01D 00
22	3.8397D-01	6.0316D-01	-6.4492D-01	6.6143D-01	2.8593D-01	1.964D-02	1.01D 00
23	4.0361D-01	5.9062D-01	-6.3203D-01	6.5070D-01	2.9765D-01	1.976D-02	1.01D 00
24	4.2337D-01	5.7826D-01	-6.1928D-01	6.4030D-01	3.0920D-01	1.987D-02	1.01D 00
25	4.4323D-01	5.6608D-01	-6.0666D-01	6.3032D-01	3.2057D-01	1.998D-02	1.01D 00
26	4.6321D-01	5.5409D-01	-5.9416D-01	6.2084D-01	3.3176D-01	2.009D-02	1.01D 00
27	4.8330D-01	5.4228D-01	-5.8178D-01	6.1196D-01	3.4277D-01	2.020D-02	1.01D 00
28	5.0350D-01	5.3065D-01	-5.6950D-01	6.0382D-01	3.5360D-01	2.030D-02	1.01D 00
29	5.2381D-01	5.1921D-01	-5.5732D-01	5.9658D-01	3.6426D-01	2.041D-02	1.01D 00
30	5.4422D-01	5.0796D-01	-5.4521D-01	5.9046D-01	3.7474D-01	2.051D-02	1.01D 00
31	5.6473D-01	4.9690D-01	-5.3315D-01	5.8574D-01	3.8505D-01	2.062D-02	1.00D 00
32	5.8535D-01	4.8603D-01	-5.2110D-01	5.8274D-01	3.9518D-01	2.072D-02	1.00D 00
33	6.0607D-01	4.7536D-01	-5.0904D-01	5.8191D-01	4.0514D-01	2.082D-02	1.00D 00
34	6.2689D-01	4.6489D-01	-4.9691D-01	5.8379D-01	4.1493D-01	2.092D-02	1.00D 00
35	6.4781D-01	4.5462D-01	-4.8465D-01	5.8910D-01	4.2455D-01	2.102D-02	1.00D 00
36	6.6883D-01	4.4456D-01	-4.7217D-01	5.9874D-01	4.3400D-01	2.113D-02	1.00D 00
37	6.8996D-01	4.3472D-01	-4.5937D-01	6.1385D-01	4.4329D-01	2.123D-02	1.01D 00
38	7.1119D-01	4.2511D-01	-4.4612D-01	6.3592D-01	4.5241D-01	2.134D-02	1.01D 00
39	7.3253D-01	4.1573D-01	-4.3224D-01	6.6685D-01	4.6138D-01	2.145D-02	1.01D 00
40	7.5398D-01	4.0662D-01	-4.1751D-01	7.0907D-01	4.7020D-01	2.156D-02	1.01D 00
41	7.7554D-01	3.9778D-01	-4.0163D-01	7.6574D-01	4.7887D-01	2.169D-02	1.01D 00
42	7.9723D-01	3.8926D-01	-3.8425D-01	8.4094D-01	4.8741D-01	2.182D-02	1.01D 00
43	8.1905D-01	3.8108D-01	-3.6487D-01	9.3998D-01	4.9581D-01	2.197D-02	1.01D 00
44	8.4101D-01	3.7330D-01	-3.4286D-01	1.0698D 00	5.0410D-01	2.213D-02	1.01D 00
45	8.6315D-01	3.6599D-01	-3.1738D-01	1.2395D 00	5.1228D-01	2.231D-02	1.01D 00
46	8.8546D-01	3.5924D-01	-2.8736D-01	1.4611D 00	5.2037D-01	2.250D-02	1.01D 00
47	9.0796D-01	3.5316D-01	-2.5137D-01	1.7502D 00	5.2838D-01	2.271D-02	1.01D 00
48	9.3067D-01	3.4793D-01	-2.0752D-01	2.1277D 00	5.3634D-01	2.293D-02	1.01D 00
49	9.5360D-01	3.4378D-01	-1.5333D-01	2.6211D 00	5.4427D-01	2.313D-02	1.01D 00
50	9.7673D-01	3.4098D-01	-8.5605D-02	3.2657D 00	5.5218D-01	2.327D-02	
51	1.0000D 00	3.3995D-01	-1.8301D-16	4.1097D 00	5.6010D-01		

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DC TODD  CDPS                        AR000346    13:45    MON AUG 07, 1978
EXAMPLE 4                            CHEMICAL DISPERSION

K =    0.00100

     NDIM      JCASE          N          NT        JOUT      LPRNT       LNORM
      250          3         51          30          20          1           1

  I1          TOL          RSC
 0.0         5.00000D-04  1.00000D-02

  A            B            C             D
 1.00000D 00 -1.00000D-03  0.0           1.00000D 00
 0.0          1.00000D 00  0.0           0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE         1           51          1          51   4.070D-01  8.590D-01  1.660D-01
NEW SPACING COMPUTED
UPDATE         2           51         48          51   3.400D-02  4.320D-01  3.480D-03
NEW SPACING COMPUTED
UPDATE         3           50         47          51   1.900D-02  6.360D-01  3.400D-03
NEW SPACING COMPUTED
UPDATE         4           51         48          51   1.050D-02  3.990D-01  5.810D-04
NEW SPACING COMPUTED
UPDATE         5           50         48          51   9.800D-03  3.280D-01  1.770D-03
UPDATE         6           51         50          51   6.370D-07  4.330D-06  5.500D-08

CONVERGED SOLUTION

CHEKDE    10  2.85D-11  0.0        0.0        2.49D-14

CHECBM     0     0     0     0  2.02D-04  4.21D-04  9.74D-02  0.0

SOLUTION WRITTEN ON UNIT 20
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DC TODD CDPS   AROC0346   13:45   MON AUG 07, 1978
EXAMPLE 4   CHEMICAL DISPERSION

	X	Y	YP	YPP	I	M	S
1	0.0	9.9887D-01	-1.1337D 00	1.4883D 00	0.0	1.676D-02	1.01D 00
2	1.6758D-02	9.8004D-01	-1.1148D 00	8.9645D-01	1.6581D-02	1.690D-02	1.01D 00
3	3.3660D-02	9.6135D-01	-1.0949D 00	1.4415D 00	3.2987D-02	1.709D-02	1.01D 00
4	5.0748D-02	9.4282D-01	-1.0762D 00	7.6489D-01	4.9256D-02	1.723D-02	1.01D 00
5	6.7974D-02	9.2443D-01	-1.0562D 00	1.5253D 00	6.5338D-02	1.742D-02	1.01D 00
6	8.5397D-02	9.0622D-01	-1.0381D 00	5.9613D-01	8.1285D-02	1.755D-02	1.01D 00
7	1.0295D-01	8.8813D-01	-1.0180D 00	1.6504D 00	9.7033D-02	1.776D-02	1.01D 00
8	1.2071D-01	8.7026D-01	-1.0004D 00	3.7906D-01	1.1265D-01	1.788D-02	1.01D 00
9	1.3859D-01	8.5249D-01	-9.8008D-01	1.8304D 00	1.2805D-01	1.811D-02	1.01D 00
10	1.5670D-01	8.3497D-01	-9.6318D-01	9.7906D-02	1.4333D-01	1.821D-02	1.01D 00
11	1.7491D-01	8.1754D-01	-9.4261D-01	2.0831D 00	1.5837D-01	1.846D-02	1.00D 00
12	1.9337D-01	8.0039D-01	-9.2661D-01	-2.6793D-01	1.7330D-01	1.853D-02	1.02D 00
13	2.1190D-01	7.8330D-01	-9.0557D-01	2.4315D 00	1.8798D-01	1.881D-02	1.00D 00
14	2.3071D-01	7.6654D-01	-8.9070D-01	-7.4532D-01	2.0255D-01	1.885D-02	1.02D 00
15	2.4956D-01	7.4979D-01	-8.6897D-01	2.9051D 00	2.1684D-01	1.917D-02	9.99D-01
16	2.6873D-01	7.3346D-01	-8.5554D-01	-1.3692D 00	2.3106D-01	1.916D-02	1.02D 00
17	2.8789D-01	7.1706D-01	-8.3281D-01	3.5417D 00	2.4496D-01	1.954D-02	9.96D-01
18	3.0743D-01	7.0118D-01	-8.2123D-01	-2.1848D 00	2.5881D-01	1.946D-02	1.02D 00
19	3.2689D-01	6.8512D-01	-7.9710D-01	4.3897D 00	2.7230D-01	1.991D-02	9.91D-01
20	3.4680D-01	6.6972D-01	-7.8786D-01	-3.2509D 00	2.8579D-01	1.974D-02	1.03D 00
21	3.6654D-01	6.5400D-01	-7.6180D-01	5.5101D 00	2.9885D-01	2.029D-02	9.86D-01
22	3.8683D-01	6.3913D-01	-7.5556D-01	-4.6436D 00	3.1197D-01	2.001D-02	1.03D 00
23	4.0684D-01	6.2372D-01	-7.2689D-01	6.9801D 00	3.2461D-01	2.068D-02	9.79D-01
24	4.2753D-01	6.0944D-01	-7.2450D-01	-6.4616D 00	3.3736D-01	2.025D-02	1.04D 00
25	4.4778D-01	5.9430D-01	-6.9231D-01	8.8962D 00	3.4955D-01	2.109D-02	9.71D-01
26	4.6886D-01	5.8068D-01	-6.9485D-01	-8.8326D 00	3.6193D-01	2.047D-02	1.05D 00
27	4.8933D-01	5.6578D-01	-6.5798D-01	1.1379D 01	3.7367D-01	2.150D-02	9.60D-01
28	5.1083D-01	5.5291D-01	-6.6685D-01	-1.1923D 01	3.8569D-01	2.065D-02	1.06D 00
29	5.3148D-01	5.3819D-01	-6.2380D-01	1.4576D 01	3.9696D-01	2.194D-02	9.47D-01
30	5.5342D-01	5.2619D-01	-6.4080D-01	-1.5947D 01	4.0863D-01	2.078D-02	1.08D 00
31	5.7421D-01	5.1156D-01	-5.8965D-01	1.8671D 01	4.1942D-01	2.240D-02	9.32D-01
32	5.9660D-01	5.0060D-01	-6.1711D-01	-2.1186D 01	4.3075D-01	2.087D-02	1.10D 00
33	6.1747D-01	4.8596D-01	-5.5541D-01	2.3886D 01	4.4104D-01	2.287D-02	9.13D-01
34	6.4034D-01	4.7629D-01	-5.9630D-01	-2.8004D 01	4.5205D-01	2.088D-02	1.12D 00
35	6.6122D-01	4.6151D-01	-5.2094D-01	3.0498D 01	4.6183D-01	2.333D-02	8.92D-01
36	6.8455D-01	4.5343D-01	-5.7908D-01	-3.6876D 01	4.7251D-01	2.081D-02	1.14D 00
37	7.0536D-01	4.3837D-01	-4.8610D-01	3.8855D 01	4.8179D-01	2.377D-02	8.68D-01
38	7.2913D-01	4.3230D-01	-5.6639D-01	-4.8414D 01	4.9214D-01	2.063D-02	1.17D 00
39	7.4976D-01	4.1680D-01	-4.5074D-01	4.9420D 01	5.0089D-01	2.411D-02	8.43D-01
40	7.7388D-01	4.1324D-01	-5.5946D-01	-6.3409D 01	5.1090D-01	2.034D-02	1.20D 00
41	7.9421D-01	3.9712D-01	-4.1460D-01	6.2847D 01	5.1914D-01	2.430D-02	8.18D-01
42	8.1851D-01	3.9566D-01	-5.5984D-01	-8.2931D 01	5.2879D-01	1.988D-02	1.22D 00
43	8.3840D-01	3.7975D-01	-3.7715D-01	8.0172D 01	5.3650D-01	2.417D-02	7.93D-01
44	8.6257D-01	3.8295D-01	-5.6946D-01	-1.0843D 02	5.4573D-01	1.918D-02	1.23D 00
45	8.8175D-01	3.6519D-01	-3.3738D-01	1.0303D 02	5.5289D-01	2.351D-02	7.77D-01
46	9.0526D-01	3.7228D-01	-5.9038D-01	-1.4174D 02	5.6157D-01	1.826D-02	1.22D 00
47	9.2352D-01	3.5360D-01	-2.9272D-01	1.3419D 02	5.6819D-01	2.229D-02	7.73D-01
48	9.4581D-01	3.6449D-01	-6.2538D-01	-1.8579D 02	5.7620D-01	1.724D-02	1.20D 00
49	9.6304D-01	3.4459D-01	-2.3771D-01	1.7868D 02	5.8230D-01	2.060D-02	7.94D-01
50	9.8364D-01	3.5888D-01	-6.7658D-01	-2.4352D 02	5.8956D-01	1.636D-02	
51	1.0000D 00	3.3744D-01	2.4869D-14	4.0803D 02	5.9524D-01		

STOP

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DC TODD  COPS                                ARO00384    19:12   MON AUG 07, 1978
EXAMPLE 5                                    INHERENTLY UNSTABLE PROBLEM

    NDIM      JCASE        N        NT       JOUT      LPRNT     LNORM
     250          1       51        30         20          1         1

  II           TOL           RSC
 0.0           5.00000D-04   1.00000D-02

  A              B             C              D
 1.00000D 00   0.0           0.0            1.00000D 00
 1.00000D 00   0.0           0.0            0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE          1          5          1          51  9.200-01   3.660 16   9.560-01
NEW SPACING COMPUTED
UPDATE          2          2          1          51  5.190-04   2.530-03   6.420-04
UPDATE          3          0          0           0  0.0        0.0        0.0

CONVERGED SOLUTION

CHEKDE    45   2.62D-12   0.0        0.0        0.0

COMPARISON OF SOLUTION WITH TABLE IN REF 7

        J         X       YA         YB         DIFF
        1       1.0     1.0000     1.0000     0.0
        2       1.2     0.7140     0.7140     0.0000
        3       1.4     0.4871     0.4869     0.0002
        4       1.6     0.3146     0.3145     0.0001
        5       1.8     0.1897     0.1897     0.0000
        6       2.0     0.1039     0.1036     0.0003
        7       2.2     0.0475     0.0476    -0.0001
        8       2.4     0.0134     0.0134    -0.0000
        9       2.6    -0.0055    -0.0058     0.0003
       10       2.8    -0.0155    -0.0153    -0.0002
       11       3.0    -0.0192    -0.0190    -0.0002
       12       3.2    -0.0192    -0.0195     0.0003
       13       3.4    -0.0181    -0.0183     0.0002
       14       4.0    -0.0131    -0.0128    -0.0003
       15       5.0    -0.0068    -0.0068    -0.0000

SOLUTION WRITTEN ON UNIT 20
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DC TODD CDPS AR000384 19:12 MON AUG 07, 1978
EXAMPLE 5 INHERENTLY UNSTABLE PROBLEM

	X	Y	YP	YPP	I	H	S
1	1.0000D 00	1.0000D 00	-1.5797D 00	1.5000D 00	0.0	2.367D-01	1.120 00
2	1.2367D 00	5.6793D-01	-1.2266D 00	1.4687D 00	1.9577D-01	2.640D-01	1.190 00
3	1.5337D 00	2.9368D-01	-8.5962D-01	1.2865D 00	3.3375D-01	3.141D-01	1.160 00
4	1.8148D 00	1.8209D-01	-5.0576D-01	9.5496D-01	4.2125D-01	3.628D-01	1.060 00
5	2.1775D 00	5.2692D-02	-2.3184D-01	5.6456D-01	4.6084D-01	3.831D-01	1.010 00
6	2.5606D 00	-2.9969D-03	-7.7838D-02	2.6179D-01	4.6850D-01	3.872D-01	1.000 00
7	2.9478D 00	-1.8442D-02	-1.3047D-02	9.3056D-02	4.6357D-01	3.875D-01	1.000 00
8	3.3353D 00	-1.8810D-02	6.7938D-03	2.1415D-02	4.5611D-01	3.875D-01	1.000 00
9	3.7229D 00	-1.5370D-02	9.6575D-03	-1.2826D-03	4.4945D-01	3.875D-01	1.000 00
10	4.1104D 00	-1.1910D-02	7.9933D-03	-5.5366D-03	4.4419D-01	3.875D-01	1.000 00
11	4.4979D 00	-9.2278D-03	5.9149D-03	-4.8047D-03	4.4012D-01	3.875D-01	1.000 00
12	4.8855D 00	-7.2629D-03	4.3177D-03	-3.4388D-03	4.3694D-01	3.875D-01	1.000 00
13	5.2730D 00	-5.8190D-03	3.2010D-03	-2.3852D-03	4.3442D-01	3.876D-01	1.000 00
14	5.6606D 00	-4.7379D-03	2.4223D-03	-1.6770D-03	4.3239D-01	3.876D-01	1.000 00
15	6.0481D 00	-3.9117D-03	1.8707D-03	-1.2067D-03	4.3072D-01	3.876D-01	1.000 00
16	6.4357D 00	-3.2685D-03	1.4687D-03	-8.8794D-04	4.2933D-01	3.876D-01	1.000 00
17	6.8232D 00	-2.7599D-03	1.1701D-03	-6.6619D-04	4.2817D-01	3.876D-01	1.000 00
18	7.2108D 00	-2.3521D-03	5.4426D-04	-5.0831D-04	4.2718D-01	3.876D-01	1.000 00
19	7.5983D 00	-2.0212D-03	7.7065D-04	-3.9360D-04	4.2633D-01	3.876D-01	1.000 00
20	7.9859D 00	-1.7498D-03	6.3536D-04	-3.0879D-04	4.2560D-01	3.876D-01	1.000 00
21	8.3734D 00	-1.5251D-03	5.2862D-04	-2.4509D-04	4.2497D-01	3.876D-01	1.000 00
22	8.7610D 00	-1.3373D-03	4.4345D-04	-1.9660D-04	4.2442D-01	3.876D-01	1.000 00
23	9.1485D 00	-1.1792D-03	3.7481D-04	-1.5921D-04	4.2393D-01	3.876D-01	1.000 00
24	9.5361D 00	-1.0451D-03	3.1898D-04	-1.3007D-04	4.2350D-01	3.876D-01	1.000 00
25	9.9237D 00	-9.3067D-04	2.7320D-04	-1.0712D-04	4.2312D-01	3.876D-01	1.000 00
26	1.0311D 01	-8.3236D-04	2.3535D-04	-8.8867D-05	4.2278D-01	3.876D-01	1.000 00
27	1.0699D 01	-7.4743D-04	2.0385D-04	-7.4232D-05	4.2247D-01	3.876D-01	1.000 00
28	1.1086D 01	-6.7369D-04	1.7746D-04	-6.2400D-05	4.2219D-01	3.876D-01	1.000 00
29	1.1474D 01	-6.0935D-04	1.5520D-04	-5.2764D-05	4.2195D-01	3.876D-01	1.000 00
30	1.1861D 01	-5.5295D-04	1.3633D-04	-4.4863D-05	4.2172D-01	3.876D-01	1.000 00
31	1.2249D 01	-5.0332D-04	1.2025D-04	-3.8341D-05	4.2152D-01	3.876D-01	1.000 00
32	1.2637D 01	-4.5945D-04	1.0647D-04	-3.2926D-05	4.2133D-01	3.876D-01	1.000 00
33	1.3024D 01	-4.2054D-04	9.4616D-05	-2.8404D-05	4.2116D-01	3.876D-01	1.000 00
34	1.3412D 01	-3.8590D-04	8.4365D-05	-2.4608D-05	4.2100D-01	3.876D-01	1.000 00
35	1.3799D 01	-3.5497D-04	7.5466D-05	-2.1405D-05	4.2086D-01	3.876D-01	1.000 00
36	1.4187D 01	-3.2726D-04	6.7711D-05	-1.8690D-05	4.2073D-01	3.876D-01	1.000 00
37	1.4574D 01	-3.0236D-04	6.0928D-05	-1.6378D-05	4.2061D-01	3.876D-01	1.000 00
38	1.4962D 01	-2.7993D-04	5.4973D-05	-1.4401D-05	4.2049D-01	3.876D-01	1.000 00
39	1.5349D 01	-2.5966D-04	4.9730D-05	-1.2703D-05	4.2039D-01	3.876D-01	1.000 00
40	1.5737D 01	-2.4130D-04	4.5095D-05	-1.1245D-05	4.2029D-01	3.876D-01	1.000 00
41	1.6124D 01	-2.2464D-04	4.0994D-05	-9.9621D-06	4.2020D-01	3.876D-01	1.000 00
42	1.6512D 01	-2.0947D-04	3.7326D-05	-8.9403D-06	4.2012D-01	3.876D-01	1.000 00
43	1.6900D 01	-1.9564D-04	3.4156D-05	-7.6642D-06	4.2004D-01	3.876D-01	1.000 00
44	1.7287D 01	-1.8301D-04	3.0911D-05	-8.1687D-06	4.1997D-01	3.876D-01	1.000 00
45	1.7675D 01	-1.7142D-04	2.9720D-05	-1.8571D-06	4.1990D-01	3.876D-01	1.000 00
46	1.8062D 01	-1.6087D-04	2.1561D-05	-2.4354D-05	4.1983D-01	3.876D-01	1.000 00
47	1.8450D 01	-1.5083D-04	4.3390D-05	7.1766D-05	4.1977D-01	3.876D-01	1.000 00
48	1.8837D 01	-1.4297D-04	-5.6499D-05	-3.2072D-04	4.1972D-01	3.876D-01	1.000 00
49	1.9225D 01	-1.3028D-04	3.4168D-04	1.2920D-03	4.1966D-01	3.876D-01	1.000 00
50	1.9612D 01	-1.3928D-04	-1.2817D-03	-5.0594D-03	4.1963D-01	3.876D-01	
51	2.0000D 01	0.0	5.7936D-03	4.7619D-02	4.1952D-01		

STOP

```
DC TODO  COPS                              AR000384     19:16    MON AUG 07, 1978
EXAMPLE 6

K =       10.0

     NDIM      JCASE          N          NT         JOUT      LPRNT      LNORM
      250          1         51          30           20          1          1

  I1           TOL          RSC
 0.0          5.00000-04   1.00000-02

  A            B            C            D
 1.00000 00   0.0          0.0          0.0
 1.00000 00   0.0          0.0          0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE         1          8          51          51  5.720-01  1.000 01  3.000-01
NEW SPACING COMPUTED
UPDATE         2         12           1          26  3.560-04  2.150-04  1.060-04

CONVERGED SOLUTION

CHEKDE    19  2.410-15  0.0       0.0       0.0

CHEKBM    12    39    12    26  2.030-04  3.420-03  2.030-02  3.090-05

SOLUTION WRITTEN ON UNIT 20
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DC TODD CDPS ARD00784 19:16 MON AUG 07, 1978
EXAMPLE 6

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-9.9971D 00	1.1974D 02	0.0	5.882D-03	1.00D 00
2	5.8820D-03	-5.5765D-02	-9.3097D 00	1.1402D 02	-1.6893D-04	5.882D-03	1.00D 00
3	1.1764D-02	-1.0958D-01	-8.6552D 00	1.0859D 02	-6.6005D-04	5.882D-03	1.18D 00
4	1.7646D-02	-1.5865D-01	-8.0317D 00	1.0245D 02	-1.4507D-03	6.913D-03	1.10D 00
5	2.4559D-02	-2.1175D-01	-7.3365D 00	9.7736D 01	-2.7338D-03	7.601D-03	1.00D 00
6	3.2161D-02	-2.6475D-01	-6.6162D 00	9.1845D 01	-4.5483D-03	7.601D-03	1.32D 00
7	3.9762D-02	-3.1244D-01	-5.9393D 00	8.6330D 01	-6.7453D-03	1.001D-02	1.01D 00
8	4.9774D-02	-3.6770D-01	-5.1092D 00	7.9587D 01	-1.0157D-02	1.009D-02	1.37D 00
9	5.9864D-02	-4.1533D-01	-4.3383D 00	7.3323D 01	-1.4114D-02	1.386D-02	1.26D 00
10	7.3729D-02	-4.6866D-01	-3.3769D 00	6.5523D 01	-2.0257D-02	1.740D-02	1.53D 00
11	9.1123D-02	-5.1797D-01	-2.3145D 00	5.6818D 01	-2.8870D-02	2.659D-02	1.75D 00
12	1.1773D-01	-5.6088D-01	-9.6353D-01	4.5427D 01	-4.3294D-02	4.663D-02	7.20D-01
13	1.6435D-01	-5.6269D-01	7.6542D-01	2.9475D 01	-6.9800D-02	3.356D-02	7.52D-01
14	1.9792D-01	-5.2219D-01	1.5960D 00	2.0198D 01	-8.8085D-02	2.523D-02	8.73D-01
15	2.2314D-01	-4.7616D-01	2.0275D 00	1.4095D 01	-1.0070D-01	2.201D-02	9.31D-01
16	2.4516D-01	-4.2851D-01	2.2839D 00	9.2703D 00	-1.1067D-01	2.049D-02	9.64D-01
17	2.6565D-01	-3.8006D-01	2.4311D 00	5.1477D 00	-1.1896D-01	1.975D-02	9.89D-01
18	2.8540D-01	-3.3128D-01	2.4961D 00	1.4879D 00	-1.2599D-01	1.954D-02	1.01D 00
19	3.0494D-01	-2.8244D-01	2.4922D 00	-1.8421D 00	-1.3198D-01	1.978D-02	1.04D 00
20	3.2471D-01	-2.3372D-01	2.4248D 00	-4.9217D 00	-1.3708D-01	2.049D-02	1.06D 00
21	3.4520D-01	-1.8528D-01	2.2939D 00	-7.8009D 00	-1.4137D-01	2.179D-02	1.10D 00
22	3.6699D-01	-1.3737D-01	2.0937D 00	-1.0508D 01	-1.4488D-01	2.394D-02	1.14D 00
23	3.9093D-01	-9.0523D-02	1.8108D 00	-1.3046D 01	-1.4759D-01	2.736D-02	1.27D 00
24	4.1829D-01	-4.6178D-02	1.4204D 00	-1.5366D 01	-1.4944D-01	3.476D-02	1.35D 00
25	4.5305D-01	-6.5620D-03	8.4683D-01	-1.7383D 01	-1.5030D-01	4.695D-02	1.00D 00
26	5.0000D-01	1.3460D-02	-1.4956D-03	-1.8392D 01	-1.4998D-01	4.695D-02	7.40D-01
27	5.4695D-01	-6.6742D-03	-8.4799D-01	-1.7394D 01	-1.4966D-01	3.476D-02	7.87D-01
28	5.8171D-01	-4.6297D-02	-1.4199D 00	-1.5378D 01	-1.5053D-01	2.736D-02	8.75D-01
29	6.0907D-01	-9.0629D-02	-1.8101D 00	-1.3056D 01	-1.5237D-01	2.394D-02	9.10D-01
30	6.3301D-01	-1.3746D-01	-2.0930D 00	-1.0517D 01	-1.5509D-01	2.179D-02	9.40D-01
31	6.5480D-01	-1.8536D-01	-2.2933D 00	-7.8082D 00	-1.5860D-01	2.049D-02	9.65D-01
32	6.7529D-01	-2.3378D-01	-2.4243D 00	-4.9279D 00	-1.6289D-01	1.978D-02	9.88D-01
33	6.9506D-01	-2.8250D-01	-2.4918D 00	-1.8473D 00	-1.6799D-01	1.954D-02	1.01D 00
34	7.1460D-01	-3.3133D-01	-2.4958D 00	1.4833D 00	-1.7399D-01	1.975D-02	1.04D 00
35	7.3435D-01	-3.8009D-01	-2.4309D 00	5.1438D 00	-1.8102D-01	2.049D-02	1.07D 00
36	7.5484D-01	-4.2855D-01	-2.2838D 00	9.2668D 00	-1.8931D-01	2.201D-02	1.15D 00
37	7.7686D-01	-4.7620D-01	-2.0276D 00	1.4092D 01	-1.9928D-01	2.523D-02	1.33D 00
38	8.0208D-01	-5.2224D-01	-1.5968D 00	2.0193D 01	-2.1189D-01	3.356D-02	1.39D 00
39	8.3565D-01	-5.6282D-01	-7.7043D-01	2.9462D 01	-2.3018D-01	4.663D-02	5.70D-01
40	8.8227D-01	-5.6128D-01	9.6183D-01	4.5387D 01	-2.5670D-01	2.659D-02	6.54D-01
41	9.0887D-01	-5.1833D-01	2.3180D 00	5.6782D 01	-2.7113D-01	1.740D-02	7.97D-01
42	9.2627D-01	-4.6896D-01	3.3809D 00	6.5493D 01	-2.7975D-01	1.386D-02	7.28D-01
43	9.4014D-01	-4.1554D-01	4.3424D 00	7.3309D 01	-2.8590D-01	1.009D-02	9.92D-01
44	9.5023D-01	-3.6789D-01	5.1133D 00	7.9567D 01	-2.8986D-01	1.001D-02	7.59D-01
45	9.6024D-01	-3.1260D-01	5.9433D 00	8.6314D 01	-2.9327D-01	7.601D-03	1.00D 00
46	9.6784D-01	-2.6487D-01	6.6202D 00	9.1833D 01	-2.9547D-01	7.601D-03	9.09D-01
47	9.7544D-01	-2.1184D-01	7.3404D 00	9.7726D 01	-2.9728D-01	6.913D-03	8.51D-01
48	9.8235D-01	-1.5871D-01	8.0356D 00	1.0344D 02	-2.9857D-01	5.882D-03	1.00D 00
49	9.8824D-01	-1.0963D-01	8.6591D 00	1.0859D 02	-2.9936D-01	5.882D-03	1.00D 00
50	9.9412D-01	-5.6788D-02	9.3136D 00	1.1401D 02	-2.9985D-01	5.882D-03	
51	1.0000D 00	0.0	1.0001D 01	1.1974D 02	-3.0002D-01		

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DC TODD  COPS                          AR000384    19:16   MON AUG 07, 1978
EXAMPLE 6

K =       15.0

     NDIM      JCASE        N          NT         JOUT      LPRNT      LNORM
      250          2        51         30          20          1          1

  I1          TOL        RSC
 0.0         5.00000-04  1.00000-02

  A           B           C           D
 1.00000 00  0.0         0.0         0.0
 1.00000 00  0.0         0.0         0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE         1          7          1          51  6.990-01  1.500 01  3.670-01
NEW SPACING COMPUTED
UPDATE         2         12          1          26  1.310-03  6.860-04  2.690-04
UPDATE         3          0          0           0  0.0       0.0       0.0

CONVERGED SOLUTION

CHEKDE   18  2.280-15  0.0       0.0       0.0

CHEKBM   12   39   12   26  9.040-04  1.500-02  2.030-01  9.550-05

SOLUTION WRITTEN ON UNIT 20
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DC TODD CDPS AR000384 19:16 MON AUG 07, 1978
EXAMPLE 6

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-1.4991D 01	2.4474D 02	0.0	4.818D-03	1.000 00
2	4.8176D-03	-6.9447D-02	-1.3850D 01	2.2905D 02	-1.6948D-04	4.818D-03	1.000 00
3	9.6351D-03	-1.3357D-01	-1.2782D 01	2.1444D 02	-6.6055D-04	4.818D-03	1.000 00
4	1.4453D-02	-1.9271D-01	-1.1783D 01	2.0083D 02	-1.4484D-03	4.818D-03	1.350 00
5	1.9270D-02	-2.4719D-01	-1.0846D 01	1.8815D 02	-2.5099D-03	6.502D-03	1.050 00
6	2.5773D-02	-3.1386D-01	-9.6747D 00	1.7239D 02	-4.3381D-03	6.803D-03	1.000 00
7	3.2576D-02	-3.7580D-01	-8.5537D 00	1.5742D 02	-6.6883D-03	6.803D-03	1.420 00
8	3.9379D-02	-4.3046D-01	-7.5299D 00	1.4386D 02	-9.4349D-03	9.629D-03	1.030 00
9	4.9008D-02	-4.9657D-01	-6.2289D 00	1.2679D 02	-1.3908D-02	9.913D-03	1.480 00
10	5.8920D-02	-5.5235D-01	-5.0504D 00	1.1150D 02	-1.9116D-02	1.463D-02	1.440 00
11	7.3548D-02	-6.1501D-01	-3.5637D 00	9.2491D 01	-2.7681D-02	2.103D-02	2.210 00
12	9.4578D-02	-6.7134D-01	-1.8713D 00	7.1017D 01	-4.1269D-02	4.650D-02	8.170-01
13	1.4107D-01	-6.9528D-01	6.1841D-01	3.9664D 01	-7.3486D-02	3.797D-02	6.970-01
14	1.7904D-01	-6.4732D-01	1.8096D 00	2.3878D 01	-9.9116D-02	2.647D-02	8.700-01
15	2.0551D-01	-5.9204D-01	2.3311D 00	1.5776D 01	-1.1555D-01	2.302D-02	9.290-01
16	2.2853D-01	-5.3473D-01	2.6258D 00	9.9693D 00	-1.2853D-01	2.138D-02	9.640-01
17	2.4991D-01	-4.7667D-01	2.7882D 00	5.3207D 00	-1.3935D-01	2.062D-02	9.910-01
18	2.7053D-01	-4.1835D-01	2.8563D 00	1.3614D 00	-1.4858D-01	2.042D-02	1.010 00
19	2.9095D-01	-3.5998D-01	2.8476D 00	-2.1484D 00	-1.5653D-01	2.070D-02	1.040 00
20	3.1165D-01	-3.0173D-01	2.7694D 00	-5.3441D 00	-1.6337D-01	2.145D-02	1.060 00
21	3.3310D-01	-2.4379D-01	2.6224D 00	-8.3063D 00	-1.6922D-01	2.277D-02	1.100 00
22	3.5587D-01	-1.8648D-01	2.4009D 00	-1.1080D 01	-1.7411D-01	2.495D-02	1.170 00
23	3.8082D-01	-1.3030D-01	2.0909D 00	-1.3684D 01	-1.7804D-01	2.927D-02	1.260 00
24	4.1010D-01	-7.5345D-02	1.6524D 00	-1.6148D 01	-1.8102D-01	3.677D-02	1.450 00
25	4.4686D-01	-2.6077D-02	1.0141D 00	-1.8304D 01	-1.8281D-01	5.314D-02	1.000 00
26	5.0000D-01	1.0913D-03	-1.5840D-03	-1.9494D 01	-1.8324D-01	5.314D-02	6.920-01
27	5.5314D-01	-2.6203D-02	-1.0148D 00	-1.8333D 01	-1.8367D-01	3.677D-02	7.960-01
28	5.8990D-01	-7.5457D-02	-1.6515D 00	-1.6173D 01	-1.8546D-01	2.927D-02	8.520-01
29	6.1918D-01	-1.3039D-01	-2.0901D 00	-1.3704D 01	-1.8844D-01	2.495D-02	9.130-01
30	6.4413D-01	-1.8655D-01	-2.4003D 00	-1.1096D 01	-1.9238D-01	2.277D-02	9.420-01
31	6.6690D-01	-2.4384D-01	-2.6220D 00	-8.3193D 00	-1.9727D-01	2.145D-02	9.650-01
32	6.8835D-01	-3.0178D-01	-2.7692D 00	-5.3556D 00	-2.0312D-01	2.070D-02	9.870-01
33	7.0905D-01	-3.6003D-01	-2.8476D 00	-2.1595D 00	-2.0997D-01	2.042D-02	1.010 00
34	7.2947D-01	-4.1840D-01	-2.8565D 00	1.3497D 00	-2.1792D-01	2.062D-02	1.040 00
35	7.5009D-01	-4.7673D-01	-2.7887D 00	5.3073D 00	-2.2714D-01	2.138D-02	1.080 00
36	7.7147D-01	-5.3480D-01	-2.6268D 00	9.9524D 00	-2.3796D-01	2.302D-02	1.150 00
37	7.9449D-01	-5.9215D-01	-2.3329D 00	1.5753D 01	-2.5095D-01	2.647D-02	1.430 00
38	8.2096D-01	-6.4750D-01	-1.8145D 00	2.3836D 01	-2.6739D-01	3.797D-02	1.220 00
39	8.5893D-01	-6.9594D-01	-6.4170D-01	3.9517D 01	-2.9303D-01	4.650D-02	4.520-01
40	9.0542D-01	-6.7297D-01	1.8761D 00	7.0651D 01	-3.2531D-01	2.103D-02	6.960-01
41	9.2645D-01	-6.1630D-01	3.5846D 00	9.2200D 01	-3.3893D-01	1.463D-02	6.780-01
42	9.4108D-01	-5.5335D-01	5.0699D 00	1.1128D 02	-3.4751D-01	9.913D-03	9.710-01
43	9.5099D-01	-4.9738D-01	6.2474D 00	1.2661D 02	-3.5272D-01	9.629D-03	7.070-01
44	9.6062D-01	-4.3110D-01	7.5473D 00	1.4371D 02	-3.5720D-01	6.803D-03	1.000 00
45	9.6742D-01	-3.7632D-01	8.5706D 00	1.5731D 02	-3.5995D-01	6.803D-03	9.560-01
46	9.7423D-01	-3.1426D-01	9.6910D 00	1.7230D 02	-3.6231D-01	6.502D-03	7.410-01
47	9.8073D-01	-2.4750D-01	1.0862D 01	1.8808D 02	-3.6414D-01	4.818D-03	1.000 00
48	9.8555D-01	-1.9294D-01	1.1798D 01	2.0078D 02	-3.6520D-01	4.818D-03	1.000 00
49	9.9036D-01	-1.3372D-01	1.2798D 01	2.1441D 02	-3.6599D-01	4.818D-03	1.000 00
50	9.9518D-01	-6.9517D-02	1.3866D 01	2.2904D 02	-3.6648D-01	4.818D-03	
51	1.0000D 00	0.0	1.5007D 01	2.4474D 02	-3.6665D-01		

```
DC TODD  COPS                         AR000384    19:16   MON AUG 07, 1978
EXAMPLE 6

K =       20.0

     NDIM      JCASE          N           NT        JOUT      LPRNT      LNORM
      250          3         51           30          20          1          1

  I1            TOL          RSC
 0.0          5.00000-04   1.00000-02

  A             B            C            D
 1.00000 00   0.0          0.0          0.0
 1.00000 00   0.0          0.0          0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE          1          7          1          51  7.740-01  2.000 01  4.000-01
NEW SPACING COMPUTED
UPDATE          2         12          1          26  2.840-03  1.450-03  5.010-04
UPDATE          3          0          0           0  0.0       0.0       0.0

CONVERGED SOLUTION

CHEKDE   33  8.020-15  0.0        0.0        0.0

CHEKBM   12   11   12   26  2.160-03  3.830-02  8.650-01  1.930-04

SOLUTION WRITTEN ON UNIT 20
```

DC TODD CDPS AR090384 19:16 MON AUG 07, 1978
EXAMPLE 6

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-1.9979D 01	4.1974D 02	0.0	4.126D-03	1.000 00
2	4.1262D-03	-7.8953D-02	-1.8712D 01	3.8808D 02	-1.6525D-04	4.126D-03	1.000 00
3	8.2524D-03	-1.5129D-01	-1.6772D 01	3.5893D 02	-6.4245D-04	4.126D-03	1.000 00
4	1.2379D-02	-2.1752D-01	-1.5247D 01	3.3207D 02	-1.4054D-03	4.126D-03	1.080 00
5	1.6505D-02	-2.7809D-01	-1.4029D 01	3.0732D 02	-2.4297D-03	4.476D-03	1.430 00
6	2.0983D-02	-3.3785D-01	-1.2710D 01	2.8268D 02	-3.8104D-03	6.411D-03	1.000 00
7	2.7392D-02	-4.1379D-01	-1.1001D 01	2.5098D 02	-6.2258D-03	6.411D-03	1.020 00
8	3.3803D-02	-4.7936D-01	-9.4836D 00	2.2306D 02	-9.0941D-03	6.542D-03	1.580 00
9	4.0345D-02	-5.3682D-01	-6.1091D 00	1.9799D 02	-1.2423D-02	1.034D-02	1.070 00
10	5.0688D-02	-6.1073D-01	-6.2421D 00	1.6440D 02	-1.8374D-02	1.109D-02	1.610 00
11	6.1783D-02	-6.7050D-01	-4.5854D 00	1.3519D 02	-2.5499D-02	1.790D-02	2.560 00
12	7.9684D-02	-7.3329D-01	-2.5433D 00	9.9453D 01	-3.8117D-02	4.591D-02	8.650-01
13	1.2559D-01	-7.7030D-01	5.8580D-01	4.6680D 01	-7.3173D-02	3.973D-02	6.860-01
14	1.6532D-01	-7.1677D-01	1.9686D 00	2.4766D 01	-1.0289D-01	2.725D-02	8.710-01
15	1.9257D-01	-6.5515D-01	2.5117D 00	1.5519D 01	-1.2162D-01	2.374D-02	9.300-01
16	2.1632D-01	-5.9174D-01	2.8057D 00	9.4643D 00	-1.3644D-01	2.207D-02	9.660-01
17	2.2838D-01	-5.2791D-01	2.5623D 00	4.8598D 00	-1.4880D-01	2.132D-02	9.930-01
18	2.5970D-01	-4.6395D-01	3.0242D 00	1.0314D 00	-1.5937D-01	2.116D-02	1.020 00
19	2.8086D-01	-3.9998D-01	3.0097D 00	-2.3394D 00	-1.6851D-01	2.150D-02	1.040 00
20	3.0237D-01	-3.3605D-01	2.9257D 00	-5.4232D 00	-1.7642D-01	2.233D-02	1.060 00
21	3.2470D-01	-2.7232D-01	2.7717D 00	-8.3136D 00	-1.8321D-01	2.376D-02	1.100 00
22	3.4846D-01	-2.0907D-01	2.5408D 00	-1.1060D 01	-1.8892D-01	2.604D-02	1.160 00
23	3.7450D-01	-1.4697D-01	2.2177D 00	-1.3676D 01	-1.9354D-01	3.015D-02	1.270 00
24	4.0465D-01	-8.6719D-02	1.7661D 00	-1.6158D 01	-1.9702D-01	3.826D-02	1.490 00
25	4.4291D-01	-3.1611D-02	1.0996D 00	-1.8400D 01	-1.9921D-01	5.709D-02	1.000 00
26	5.0000D-01	7.6750D-05	-1.4328D-03	-1.9709D 01	-1.9981D-01	5.709D-02	6.700-01
27	5.5709D-01	-3.1726D-02	-1.0999D 00	-1.8446D 01	-2.0041D-01	3.826D-02	7.880-01
28	5.9535D-01	-8.6806D-02	-1.7650D 00	-1.6193D 01	-2.0260D-01	3.015D-02	8.640-01
29	6.2550D-01	-1.4703D-01	-2.2169D 00	-1.3700D 01	-2.0609D-01	2.604D-02	9.130-01
30	6.5154D-01	-2.0911D-01	-2.5403D 00	-1.1078D 01	-2.1071D-01	2.376D-02	9.400-01
31	6.7530D-01	-2.7226D-01	-2.7715D 00	-8.3277D 00	-2.1642D-01	2.233D-02	9.630-01
32	6.9763D-01	-3.3609D-01	-2.9257D 00	-5.4367D 00	-2.2320D-01	2.150D-02	9.840-01
33	7.1914D-01	-4.0002D-01	-3.0101D 00	-2.3544D 00	-2.3111D-01	2.116D-02	1.010 00
34	7.4030D-01	-4.6400D-01	-3.0249D 00	1.0122D 00	-2.4026D-01	2.132D-02	1.040 00
35	7.6162D-01	-5.2798D-01	-2.9635D 00	4.8329D 00	-2.5083D-01	2.207D-02	1.080 00
36	7.8368D-01	-5.9184D-01	-2.8078D 00	9.4232D 00	-2.6319D-01	2.374D-02	1.150 00
37	8.0743D-01	-6.5532D-01	-2.5155D 00	1.5451D 01	-2.7801D-01	2.725D-02	1.460 00
38	8.3468D-01	-7.1712D-01	-1.9797D 00	2.4626D 01	-2.9675D-01	3.973D-02	1.160 00
39	8.7441D-01	-7.7182D-01	-6.4308D-01	4.6071D 01	-3.2650D-01	4.591D-02	3.900-01
40	9.2032D-01	-7.3692D-01	2.5712D 00	9.8001D 01	-3.6169D-01	1.790D-02	6.200-01
41	9.3822D-01	-6.7321D-01	4.6462D 00	1.3411D 02	-3.7427D-01	1.109D-02	9.320-01
42	9.4931D-01	-6.1282D-01	6.2928D 00	1.6356D 02	-3.8152D-01	1.034D-02	6.330-01
43	9.5965D-01	-5.3841D-01	8.1550D 00	1.9775D 02	-3.8749D-01	6.542D-03	9.800-01
44	9.6620D-01	-4.8066D-01	9.5271D 00	2.2254D 02	-3.9083D-01	6.411D-03	1.000 00
45	9.7261D-01	-4.1482D-01	1.1042D 01	2.5057D 02	-3.9370D-01	6.411D-03	6.980-01
46	9.7902D-01	-3.3866D-01	1.2748D 01	2.8237D 02	-3.9613D-01	4.476D-03	9.220-01
47	9.8350D-01	-2.7870D-01	1.4067D 01	3.0708D 02	-3.9751D-01	4.126D-03	1.000 00
48	9.8762D-01	-2.1797D-01	1.5384D 01	3.3189D 02	-3.9854D-01	4.126D-03	1.000 00
49	9.9175D-01	-1.5159D-01	1.6809D 01	3.5881D 02	-3.9930D-01	4.126D-03	1.000 00
50	9.9587D-01	-7.9101D-02	1.8349D 01	3.8802D 02	-3.9978D-01	4.126D-03	
51	1.0000D 00	0.0	2.0015D 01	4.1974D 02	-3.9994D-01		

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DC TODD  CDPS                          AR000384    19:16   MON AUG 07, 1978
EXAMPLE 6

K =       25.0

     NDIM      JCASE          N          NT        JOUT      LPRNT      LNORM
      250          4         51          30          20          1          1

  11          TOL          ASC
 0.0         5.00000D-04  1.00000D-02

  A           B            C            D
 1.00000D 00  0.0          0.0          0.0
 1.00000D 00  0.0          0.0          0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE         1         6          1          51  8.22D-01  2.51D 01  4.20D-01
NEW SPACING COMPUTED
UPDATE         2        12          1          26  3.68D-03  2.33D-03  5.93D-04
UPDATE         3         0          0           0  0.0       0.0       0.0

CONVERGED SOLUTION

CHEKDE   18  5.37D-14  0.0        0.0        0.0

CHEKBM   12   11   12   26  2.97D-03  6.01D-02  1.86D 00  2.35D-04

SOLUTION WRITTEN ON UNIT 20
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DC TODD CDPS
EXAMPLE 6

AR000384 19:16 MON AUG 07, 1978

	X	Y	YP	YPP	T	H	S
1	0.0	0.0	-2.4969D 01	6.4474D 02	0.0	3.641D-03	1.000 00
2	3.6412D-03	-8.6765D-02	-2.2721D 01	5.9042D 02	-1.6045D-04	3.641D-03	1.000 00
3	7.2824D-03	-1.6570D-01	-2.0663D 01	5.4083D 02	-6.2236D-04	3.641D-03	1.000 00
4	1.0924D-02	-2.3745D-01	-1.8778D 01	4.9555D 02	-1.3584D-03	3.641D-03	1.000 00
5	1.4565D-02	-3.0263D-01	-1.7050D 01	4.5420D 02	-2.3436D-03	3.641D-03	1.360 00
6	1.8206D-02	-3.6175D-01	-1.5466D 01	4.1645D 02	-3.5550D-03	4.962D-03	1.260 00
7	2.3168D-02	-4.3360D-01	-1.3517D 01	3.7023D 02	-5.5323D-03	6.244D-03	1.000 00
8	2.9412D-02	-5.1113D-01	-1.1368D 01	3.1963D 02	-8.4888D-03	6.244D-03	1.240 00
9	3.5656D-02	-5.7617D-01	-9.5110D 00	2.7633D 02	-1.1890D-02	7.753D-03	1.440 00
10	4.3409D-02	-6.4209D-01	-7.5520D 00	2.3115D 02	-1.6622D-02	1.120D-02	1.510 00
11	5.4611D-02	-7.1336D-01	-5.2657D 00	1.7952D 02	-2.4238D-02	1.687D-02	2.560 00
12	7.1484D-02	-7.7999D-01	-2.7997D 00	1.2429D 02	-3.6894D-02	4.318D-02	9.300-01
13	1.1466D-01	-8.1776D-01	6.1986D-01	5.1082D 01	-7.1909D-02	4.014D-02	6.960-01
14	1.5480D-01	-7.6045D-01	2.0676D 00	2.4300D 01	-1.0378D-01	2.793D-02	8.670-01
15	1.8273D-01	-6.9461D-01	2.6029D 00	1.4651D 01	-1.2413D-01	2.421D-02	9.390-01
16	2.0694D-01	-6.2792D-01	2.8842D 00	8.8541D 00	-1.4015D-01	2.273D-02	9.700-01
17	2.2967D-01	-5.6046D-01	3.0350D 00	4.5422D 00	-1.5366D-01	2.205D-02	9.930-01
18	2.5171D-01	-4.9275D-01	3.0948D 00	9.5663D-01	-1.6527D-01	2.190D-02	1.010 00
19	2.7361D-01	-4.2502D-01	3.0801D 00	-2.2401D 00	-1.7532D-01	2.220D-02	1.030 00
20	2.9581D-01	-3.5744D-01	2.9969D 00	-5.2127D 00	-1.8400D-01	2.296D-02	1.070 00
21	3.1877D-01	-2.9026D-01	2.8443D 00	-8.0431D 00	-1.9143D-01	2.448D-02	1.100 00
22	3.4325D-01	-2.2333D-01	2.6131D 00	-1.0789D 01	-1.9771D-01	2.693D-02	1.140 00
23	3.7018D-01	-1.5721D-01	2.2857D 00	-1.3456D 01	-2.0281D-01	3.076D-02	1.280 00
24	4.0094D-01	-9.3691D-02	1.8311D 00	-1.5987D 01	-2.0663D-01	3.934D-02	1.520 00
25	4.4028D-01	-3.4728D-02	1.1505D 00	-1.8326D 01	-2.0907D-01	5.972D-02	1.000 00
26	5.0000D-01	-5.1352D-06	-1.2607D-03	-1.9742D 01	-2.0977D-01	5.972D-02	6.590-01
27	5.5972D-01	-3.4830D-02	-1.1505D 00	-1.8390D 01	-2.1046D-01	3.934D-02	7.820-01
28	5.9906D-01	-9.3757D-02	-1.8300D 00	-1.6029D 01	-2.1290D-01	3.076D-02	8.750-01
29	6.2982D-01	-1.5725D-01	-2.2850D 00	-1.3479D 01	-2.1673D-01	2.693D-02	9.090-01
30	6.5675D-01	-2.2336D-01	-2.6128D 00	-1.0804D 01	-2.2183D-01	2.448D-02	9.380-01
31	6.8123D-01	-2.9028D-01	-2.8441D 00	-8.0538D 00	-2.2811D-01	2.296D-02	9.670-01
32	7.0419D-01	-3.5746D-01	-2.9970D 00	-5.2224D 00	-2.3554D-01	2.220D-02	9.860-01
33	7.2639D-01	-4.2504D-01	-3.0804D 00	-2.2514D 00	-2.4422D-01	2.190D-02	1.010 00
34	7.4829D-01	-4.9278D-01	-3.0953D 00	9.4045D-01	-2.5427D-01	2.205D-02	1.030 00
35	7.7033D-01	-5.6051D-01	-3.0360D 00	4.5160D 00	-2.6588D-01	2.273D-02	1.060 00
36	7.9306D-01	-6.2800D-01	-2.8862D 00	8.8076D 00	-2.7940D-01	2.421D-02	1.150 00
37	8.1727D-01	-6.9476D-01	-2.6069D 00	1.4562D 01	-2.9542D-01	2.793D-02	1.440 00
38	8.4520D-01	-7.6082D-01	-2.0823D 00	2.4072D 01	-3.1578D-01	4.014D-02	1.080 00
39	8.8534D-01	-8.1985D-01	-7.0618D-01	4.9776D 01	-3.4768D-01	4.318D-02	3.910-01
40	9.2852D-01	-7.8494D-01	2.8470D 00	1.2120D 02	-3.8288D-01	1.687D-02	6.640-01
41	9.4539D-01	-7.1692D-01	5.3588D 00	1.7730D 02	-3.9561D-01	1.120D-02	6.920-01
42	9.5659D-01	-6.4471D-01	7.6290D 00	2.2951D 02	-4.0326D-01	7.753D-03	8.050-01
43	9.6434D-01	-5.7822D-01	9.5810D 00	2.7505D 02	-4.0801D-01	6.244D-03	1.000 00
44	9.7059D-01	-5.1276D-01	1.1431D 01	3.1861D 02	-4.1142D-01	6.244D-03	7.950-01
45	9.7683D-01	-4.3485D-01	1.3576D 01	3.6944D 02	-4.1438D-01	4.962D-03	7.340-01
46	9.8179D-01	-3.6276D-01	1.5522D 01	4.1584D 02	-4.1637D-01	3.641D-03	1.000 00
47	9.8544D-01	-3.0340D-01	1.7104D 01	4.5372D 02	-4.1758D-01	3.641D-03	1.000 00
48	9.8908D-01	-2.3802D-01	1.8831D 01	4.9519D 02	-4.1857D-01	3.641D-03	1.000 00
49	9.9272D-01	-1.6607D-01	2.0715D 01	5.4060D 02	-4.1931D-01	3.641D-03	1.000 00
50	9.9636D-01	-8.6953D-02	2.2773D 01	5.9031D 02	-4.1977D-01	3.641D-03	
51	1.0000D 00	0.0	2.5021D 01	6.4474D 02	-4.1993D-01		

STOP

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DC TODD  CDPS                              AH000384    19:29   MON AUG 07, 1978
EXAMPLE 7                                  Y'' = EXP(Y)

J =    19      LAMBDA =    8.92522409945520-01                  40E47C5941A9C251     40E47C5941A9C251

     NDIM       JCASE          N          NT        JOUT       LPRNT      LNORM
      250          1          51          30          20           1          1

  I1          TOL          RSC
 0.0          5.00000-04  1.00000-02

  A           B            C            D
 1.00000 00  0.0          0.0          0.0
 1.00000 00  0.0          0.0          0.0

UNIFORM SPACING COMPUTED

SOLUTION GUESSED

UPDATE         1         26          51          51   1.130-01  4.620-01  7.580-02
NEW SPACING COMPUTED
UPDATE         2         26          50          51   4.620-03  1.640-03  4.300-03
UPDATE         3         26          50          51   8.540-08  2.940-08  7.860-08

CONVERGED SOLUTION

CHEKOE    28  2.330-16  0.0         0.0         0.0

CHEKBM    50    27    40     0  2.170-07  1.050-06  4.730-09  0.0

SOLUTION WRITTEN ON UNIT 20
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DC TODD COPS
EXAMPLE 7

AR000384 19:20 MON AUG 07, 1978
Y'' = EXP(Y)

	X	Y	YP	YPP	I	H	S
1	0.0	0.0	-4.6363D-01	1.0000D 00	0.0	1.883D-02	1.01D 00
2	1.8833D-02	-8.5547D-03	-4.4488D-01	9.9148D-01	-8.1108D-05	1.896D-02	1.01D 00
3	3.7796D-02	-1.6813D-02	-4.2616D-01	9.8333D-01	-3.2219D-04	1.909D-02	1.01D 00
4	5.6885D-02	-2.4770D-02	-4.0746D-01	9.7553D-01	-7.1966D-04	1.921D-02	1.01D 00
5	7.6099D-02	-3.2419D-02	-3.8879D-01	9.6810D-01	-1.2696D-03	1.933D-02	1.01D 00
6	9.5434D-02	-3.9755D-02	-3.7014D-01	9.6102D-01	-1.9679D-03	1.945D-02	1.01D 00
7	1.1488D-01	-4.6774D-02	-3.5151D-01	9.5430D-01	-2.8101D-03	1.956D-02	1.01D 00
8	1.3445D-01	-5.3468D-02	-3.3291D-01	9.4794D-01	-3.7912D-03	1.967D-02	1.01D 00
9	1.5412D-01	-5.9834D-02	-3.1432D-01	9.4192D-01	-4.9063D-03	1.978D-02	1.01D 00
10	1.7390D-01	-6.5867D-02	-2.9575D-01	9.3626D-01	-6.1499D-03	1.988D-02	1.00D 00
11	1.9377D-01	-7.1560D-02	-2.7719D-01	9.3094D-01	-7.5162D-03	1.997D-02	1.00D 00
12	2.1374D-01	-7.6911D-02	-2.5865D-01	9.2597D-01	-8.9994D-03	2.006D-02	1.00D 00
13	2.3380D-01	-8.1913D-02	-2.4012D-01	9.2135D-01	-1.0593D-02	2.014D-02	1.00D 00
14	2.5395D-01	-8.6563D-02	-2.2161D-01	9.1708D-01	-1.2290D-02	2.022D-02	1.00D 00
15	2.7417D-01	-9.0857D-02	-2.0310D-01	9.1315D-01	-1.4085D-02	2.029D-02	1.00D 00
16	2.9446D-01	-9.4791D-02	-1.8461D-01	9.0956D-01	-1.5969D-02	2.036D-02	1.00D 00
17	3.1482D-01	-9.8362D-02	-1.6612D-01	9.0632D-01	-1.7936D-02	2.042D-02	1.00D 00
18	3.3524D-01	-1.0157D-01	-1.4765D-01	9.0342D-01	-1.9978D-02	2.047D-02	1.00D 00
19	3.5572D-01	-1.0440D-01	-1.2918D-01	9.0087D-01	-2.2087D-02	2.052D-02	1.00D 00
20	3.7624D-01	-1.0686D-01	-1.1071D-01	8.9865D-01	-2.4256D-02	2.056D-02	1.00D 00
21	3.9680D-01	-1.0895D-01	-9.2252D-02	8.9678D-01	-2.6475D-02	2.060D-02	1.00D 00
22	4.1740D-01	-1.1066D-01	-7.3797D-02	8.9525D-01	-2.8738D-02	2.063D-02	1.00D 00
23	4.3803D-01	-1.1199D-01	-5.5345D-02	8.9405D-01	-3.1034D-02	2.065D-02	1.00D 00
24	4.5867D-01	-1.1294D-01	-3.6895D-02	8.9320D-01	-3.3357D-02	2.066D-02	1.00D 00
25	4.7933D-01	-1.1351D-01	-1.8447D-02	8.9269D-01	-3.5697D-02	2.067D-02	1.00D 00
26	5.0000D-01	-1.1370D-01	-1.9327D-08	8.9252D-01	-3.8046D-02	2.067D-02	1.00D 00
27	5.2067D-01	-1.1351D-01	1.8447D-02	8.9269D-01	-4.0394D-02	2.066D-02	9.99D-01
28	5.4133D-01	-1.1294D-01	3.6895D-02	8.9320D-01	-4.2734D-02	2.065D-02	9.99D-01
29	5.6197D-01	-1.1199D-01	5.5345D-02	8.9405D-01	-4.5057D-02	2.063D-02	9.99D-01
30	5.8260D-01	-1.1066D-01	7.3797D-02	8.9525D-01	-4.7354D-02	2.060D-02	9.98D-01
31	6.0320D-01	-1.0895D-01	9.2252D-02	8.9678D-01	-4.9616D-02	2.056D-02	9.98D-01
32	6.2376D-01	-1.0686D-01	1.1071D-01	8.9865D-01	-5.1835D-02	2.052D-02	9.98D-01
33	6.4428D-01	-1.0440D-01	1.2918D-01	9.0087D-01	-5.4004D-02	2.047D-02	9.97D-01
34	6.6476D-01	-1.0157D-01	1.4765D-01	9.0342D-01	-5.6113D-02	2.042D-02	9.97D-01
35	6.8518D-01	-9.8362D-02	1.6612D-01	9.0632D-01	-5.8155D-02	2.036D-02	9.97D-01
36	7.0554D-01	-9.4791D-02	1.8461D-01	9.0956D-01	-6.0122D-02	2.029D-02	9.96D-01
37	7.2583D-01	-9.0857D-02	2.0310D-01	9.1315D-01	-6.2006D-02	2.022D-02	9.96D-01
38	7.4605D-01	-8.6563D-02	2.2161D-01	9.1708D-01	-6.3801D-02	2.014D-02	9.96D-01
39	7.6620D-01	-8.1913D-02	2.4012D-01	9.2135D-01	-6.5498D-02	2.006D-02	9.96D-01
40	7.8626D-01	-7.6911D-02	2.5865D-01	9.2597D-01	-6.7092D-02	1.997D-02	9.95D-01
41	8.0623D-01	-7.1560D-02	2.7719D-01	9.3094D-01	-6.8575D-02	1.988D-02	9.95D-01
42	8.2610D-01	-6.5867D-02	2.9575D-01	9.3626D-01	-6.9941D-02	1.978D-02	9.95D-01
43	8.4588D-01	-5.9835D-02	3.1432D-01	9.4192D-01	-7.1185D-02	1.967D-02	9.94D-01
44	8.6555D-01	-5.3468D-02	3.3291D-01	9.4794D-01	-7.2300D-02	1.956D-02	9.94D-01
45	8.8512D-01	-4.6774D-02	3.5151D-01	9.5430D-01	-7.3281D-02	1.945D-02	9.94D-01
46	9.0457D-01	-3.9755D-02	3.7014D-01	9.6102D-01	-7.4123D-02	1.933D-02	9.94D-01
47	9.2390D-01	-3.2419D-02	3.8879D-01	9.6810D-01	-7.4822D-02	1.921D-02	9.94D-01
48	9.4311D-01	-2.4770D-02	4.0746D-01	9.7553D-01	-7.5372D-02	1.909D-02	9.93D-01
49	9.6220D-01	-1.6813D-02	4.2616D-01	9.8333D-01	-7.5769D-02	1.896D-02	9.93D-01
50	9.8117D-01	-8.5547D-03	4.4488D-01	9.9148D-01	-7.6010D-02	1.883D-02	
51	1.0000D 00	0.0	4.6363D-01	1.0000D 00	-7.6091D-02		

STOP

NOMENCLATURE

a_{1i}	Eq. (104)
a_{2i}	Eq. (105)
A_{ij}, i = 1 and N, j = 1 and N	Eqs. (B-26) thru (B-29)
A_i, B_i, C_i, D_i, i = 1 and N	End conditions parameters, Eq. (24)
d_{imjk}	System coefficients, Eqs. (111) thru (133)
D	Differentiation operator, Eq. (40)
D_{mi}	Eq. (110)
E	Eq. (B-4)
E_i	Operator, Eq. (38)
F	Eq. (B-5)
$f(x, y, y', I)$	Eq. (134)
g_i	Integral of quintic spline at x_i, Eqs. (95) and (96)
H	Eq. (B-1)
h_i	Length of the ith interval, Eq. (1)
I	Integral of y(x), Eq. (93)
I_i	Value of I at x_i
$J_i(a,b)$	Integral of the cubic spline, Eq. (27)
$K_i(a,b)$	Integral of the quintic spline, Eq. (89)
k	Parameter in example problems, Eqs. (146), (152), (159), and (169)
M_i	Second derivative of the cubic spline at x_i, Eq. (8)

$\widetilde{M}$	Eq. (B-3)
m_i	First derivative of the cubic spline at x_i, Eq. (7)
$\widetilde{m}$	Eq. (B-2)
m_i	First derivative of the quintic spline at x_i, Eqs. (74) and (65)
$\mathfrak{M}_i$	Second derivative of the quintic spline at x_i, Eqs. (75) and (59)
N	Number of points
$p_i(x)$	Cubic spline polynomial in the ith interval, Eqs. (5) and (11)
Q_i	Eq. (98)
$q_i(x)$	Quintic spline polynomial in the ith interval, Eqs. (73) and (86)
R_i	Eq. (109)
$x(x)$	Cubic spline, Eq. (5)
V_i	Eq. (108)
x	Independent variable
x_i	Given abscissas
y	Dependent varaible
y_{Ai}, y_{Bi}	Used in definition of $p_i(x)$, Eqs. (9) and (10)
y_{ai}, y_{bi}, y_{ci}, y_{di}	Used in the definition of $q_i(x)$, Eqs. (82) thru (85)
y_i	Given ordinates or else $y(x_i)$
$z(x)$	Quintic spline, Eq. (73)
z_i	$z(x_i)$
$\alpha(x)$, $\beta(x)$, $\lambda(x)$, $\epsilon(x)$	Coefficients of linear or linearized differential equation, Eqs. (92) and (136) thru (139)
δ_i	Parameter used in definition of m_i, Eq. (66)

Δ_i	Parameter used in definition of $\mathfrak{M}_i$, Eq. (60)
η_i	Eq. (3)
θ_i	Eq. (4)
λ	Eqs. (153) and (177)
λ_i, i = 1 and $\dot{N}$	Lagrange multipliers, Eq. (B-5)
σ_i	Eq. (2)